Limit theorems for stationary processes

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soutenue à Tours le : 25 juin 2012

pour obtenir le grade de : Docteur de l'université François - Rabelais de Tours

Discipline : Mathématiques

LES THÉORÈMES LIMITES POUR DES PROCESSUS STATIONNAIRES

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Remerciements

Cette thèse n'aurait pas été possible sans l'aide de nombreuses personnes. Tout d'abord, je tiens à remercier mon directeur de thèse Monsieur Jérôme Depauw. En fait, je n'aurais pas pu terminer la thèse sans son précieuse aide. Encore une fois, je tiens à le remercier pour son aide.

I would like to thank Mr. Tran Loc Hung, my thesis co-advisor, for helping me during the period I was staying in Vietnam.

Je tiens à remercier Monsieur Yves Derriennic et Monsieur Olivier Garet pour avoir consacré leur precieux temps de lire, corriger et juger mon travail de thèse.

Je suis honoré que Monsieur Dalibor Volny, Monsieur Pierre Andreoletti et Monsieur Marc Peigné aient accepté de faire partie de mon jury de thèse.

Ensuite, je tiens à remercier Le Pôle Universitaire Français (PUF) à Ho Chi Minh ville (Vietnam), et Monsieur Michel Zinsmeister (l'université d'Orléans), qui ont creé des occasions et ils ont fourni des fonds pour mon programme de doctorat. Je tiens aussi remercier Monsieur Emmanuel Lesigne, directeur du Laboraroire de Mathématiques et Physique Théorique (LMPT) et Monsieur Guy Barles, directeur de Fédération Denis Poisson (FDP) pour le financement partiel pour mes études en France. Par ailleurs, je remercie aussi Le Formath-Vietnam qui a également appuyé le financement de ma thèse.

Je tiens à remercie tous les membres du LMPT pour leur chaleureux accueil et leur aide. En particulier, je tiens à remercier Sandrine Renard-Riccetti, Anne-Marie Chenais-Kermorvant, Bernadette Valle, Anouchka Lepine, Nguyen Phuoc Tai, Safaa El Sayed, Dao Nguyen Anh, Nguyen Quoc Hung,...

À l'université de Cantho où je travaille, je tiens à remercier mes collègues à la faculté des sciences. Ils m'ont toujours encouragé et aidé pendant mon processus d'apprentissage. Dac biet, tôi xin duoc bay to long biet on sau sac den cô Tran Ngoc Lien, nguoi luon quan tam den viec hoc cua tôi va luon danh cho tôi nhung tinh cam that tham tinh va cao ca ngay tu nhung ngay dau tien tôi duoc vao lam viec o khoa Khoa Hoc. Je remercie aussi mes amis: Do Minh Khang, Nguyen Huynh Nhu, Nguyen Kim Ngan, Le Pham Ai Tam, Nguyen Khanh Van,... pour leurs partages. Ils m'ont toujours fait plaisir après des moments durs d'tudes.

Enfin, je tiens à remercier en particulier ma famille, mes parents, mon frère ainé, mon jeune frère et ma jeune soeur. Je suis toujours très heureux quand je pense à eux.

Merci à toutes et à tous !

Résumé

Nous étudions la mesure spectrale des transformations stationnaires, puis nous l'utilisons pour étudier le théorème ergodique et le théorème limite central. Nous étudions également les martingales avec une nouvelle preuve du théorème central limite, sans analyse de Fourier. Pour le théorème limite central pour marches aléatoires dans un environnement aléatoire sur la dimension 1, on donne deux méthodes pour l'obtenir: approximation pour une martingale et méthode des moments. La méthode des martingales fait résoudre l'equation de Dirichlet $(I-P)h = 0$, alors que celle des moments résoudre l'equation de Poisson $(I - P)h = f$. Enfin, nous pouvons utiliser la deuxième méthode pour prouver la relation d'Einstein pour des diffusions réversibles dans un environnement aléatoire dans une dimension.

Mots clés : mesure spectrale, théoréme limite centrale pour martingale, martingale approximation, marche aléatoire dans un environnement aléatoire, la relation d'Einstein.

Abstract

We study the spectral measure for stationary transformations, and then apply to Ergodic theorem and Central limit theorem. We study also martingale process with a new proof of the central limit theorem without Fourier analysis. For the central limit theorem for random walks in random environment, we give two methods to obtain it: martingale approximation and moments. The method of martingales solves Dirichlet's equation $(I-P)h=0$, and the method of moments solves Poisson's equation $(I-P)h=f$. Finally, we can use the second method to prove the Einstein relation for reversible diffusions in random environment in one dimension.

Keywords : spectral measure, martingale central limit theorem, martingale approximation, random walk in random environment, Einstein's relation.

Contents

Introduction

La mesure spectrale des transformations stationnaires associées à une fonction est bien connue. Pour l'application au théorème central limite, en 1986, Kipnis et Varadhan [29] ont donné une condition nécessaire (1.25) pour obtenir le théorème central limite dans le contexte des chaines réversibles par resolution de l'équation de Poisson via la résolvante. Dans la suite, nous allons construire à nouveau la mesure spectrale pour une transformation inversible ou réversible de la chaine de Markov et ensuite l'appliquer au théorème ergodique et au théorème central limite. Le théorème de Kipnis et Varadhan [29] est considéré comme un exemple intéressant. Nous étudions également la mesure spectrale avec des valeurs dans l'espace de l'opérateur.

Initié avec un résultat de Billingsley [2], Ibragimov [26] et ensuite Brown [8], le théorème limite central pour les martingales a été étudié et très bien développés jusqu' à pérsent (voir Hall & Heyde [23]). Dans leur preuve, ces auteurs utilisent la fonction caractéristique. Dans cette thèse, nous allons étudier une nouvelle méthode pour le théorème central limite, surtout pour martingale, sans utiliser l'analyse de Fourier. Le point de cette méthode est d'utiliser le developpement de Taylor à l'ordre 2 de la fonction f appartenant à $\mathcal{C}_K^2$, combiné des idées adaptées de Linderberg ([36], 1922), Trotter ([48], 1959), Billingsley ([2], 1961), Brown ([8], 1971).

Le théorème limite central pour la marche aléatoire sur un réseau stationnaire de conductances a été étudié par plusieurs auteurs. En une dimension, lorsque conductances et les résistances sont intégrables, une méthode de martingale introduite par S. Kozlov ([31], 1985) permet de prouver le théorème limite centrale "Quenched". Dans ce cas, la variance de la loi limite n'est pas nulle. Si les résistances ne sont pas intégrables, le théorème limite centrale "Annealed" avec une variance nulle a été établie par Y. Derriennic et M. Lin (communication personnelle). Et puis, dans un document de J. Depauw et J-M. Derrien ([12], 2009), ils ont prouvé la version Quenched de la convergence de la variance par une méthode simple qui utilise le théorème ergodique ponctuel (voir [51]), sans utiliser aucune martingale. Nous avons deux méthodes pour établir le théorème de la limite centrale Quenched pour la marche aléatoire réversible en milieu aléatoire sur $\mathbb{Z}$. La première méthode est d'utiliser l'approximation par une martingale et le seconde est d'adapter J. Depauw et J-M. Derrien [12] sans utiliser aucune martingale. Pour la diffusion en continu, le théorème de la limite centrale Quenched pour le temp continu et l'espace discret sera montré en détail par un moyen similaire. Enfin, nous prouvons la relation d'Einstein pour des diffusions réversibles dans un environnement aléatoire dans une dimension.

Cette thèse est organisée comme suit:

Chapitre 1: On construit à nouveau la mesure spectrale des transformations stationnaires associées à une fonction dans L^2 et ensuite nous donnons quelques exemples de leurs applications pour le théorème ergodique et le théorème central limite pour les chaines de Markov réversibles. La preuve du théorème de Kipnis-Varadhan (1986) est montré en détail. Nous rappelons aussi à la mesure spectrale avec des valeurs dans l'espace de l'opérateur.

Chapitre 2: Nous donnons une nouvelle méthode pour obtenir le TLC pour les cas d'indépendance des variables et des processus de martingale. Le point de cette méthode est d'utiliser le developpement de Taylor à l'ordre 2 de la fonction f appartenant à $\mathcal{C}_K^2$, combinée à une technique nouvelle et des idées adaptées de Trotter (1959), Billingsley (1961), Brown (1971),...

Chapitre 3: Les théorèmes de Gordin-Kipnis pour les fonctionnels addives de chaines de Markov stationnaire et puis pour la chaine de Markov partant d'un point sont passés en revue. Ces théorèmes sont très classiques, mais nous détaillons les épreuves avec soin, parce que ils sont très utiles pour la convergence des marches aléatoires dans un environnement aléatoire dans les chapitres suivants.

Chapitre 4: Ce chapitre est consacré à le TLC pour les marches aléatoires dans un environnement aléatoire sur $\mathbb{Z}$. Le TLC pour les marches alèatoires sera valide si la fonction mesurable c définie sur Ω, l'espace des environnements, associée à la conductivité de l'arête et de son inverse appartiennent à L^1. L'approximation par une martingale est utilisé dans la preuve, adaptée de Boivin (1993).

Chapitre 5: L'objectif principal de ce chapitre est d'obtenir le TLC pour les marches aléatoires dans un environnement aléatoire dans le chapitre 4 sans martingales. Plus précisément, la convergence est fondée sur les moments des variables. Un analogue en temps continu et espace discret est donné.

Chapitre 6: Nous considérons la relation d'Einstein pour les marches aléatoires dans un environnement aléatoire par la même méthode que dans le chapitre précédent. Supposons qu'il y a une dérive $\lambda \neq 0$, nous allons étudier la convergence de léspérance de la marche aléatoire lorsque la "drift" λ tend vers zéro.

Introduction

The spectral measure for stationary transformations associated to a function is well-known. For the application to central limit theorem, in 1986 Kipnis and Varadhan [29] gave a necessary condition (1.25) to obtain the Central limit theorem in the context of reversible chains by solving the Poisson equation approximately via the resolvent. In the sequel, we will build again the spectral measure for invertible transformation and reversible Markov chain and then apply to Ergodic theorem and Central limit theorem. The theorem of Kipnis and Varadhan [29] is regarded as an interesting example. We study also the spectral measure with values in operator's space.

Starting with a result of Billingsley [2], Ibragimov [26] and then Brown [8], the limit theorey for martingales has been studied and very well-developed up to now (see Hall & Heyde [23]). In their proof, they use characteristic fuction to obtain the limit. In this thesis, we will study a new method for the central limit theorem, especially for martinggale, without using Fourier analysis. The point of this method is to use Taylor's expansion of function f belongs to $\mathcal{C}_K^2$, combined some ideas adapted from Linderberg ([36], 1922), Trotter ([48], 1959), Billingsley ([2], 1961), Brown ([8], 1971).

The Central limit theorem for random walk on a stationary network of conductances has been studied by several authors. In one dimension, when conductances and resistances are integrable, and following a method of martingale introduced by S. Kozlov ([31], 1985), we can prove the Quenched Central limit theorem. In that case the variance of the limit law is not null. When resistances are not integrable, the Annealed Central limit theorem with null variance was established by Y. Derriennic and M. Lin (personal communication). And then, in a paper of J. Depauw and J-M. Derrien ([12], 2009), they proved the quenched version to obtain the limit of the variance by a simple method that is using the pointwise ergodic theorem (see [51]) in their proof and without using any martingale. In this works, we will two methods to establish the Quenched Central limit theorem for reversible random walk in random environment on $\mathbb{Z}$. The first method is using martingale approximation and the second one is to adapt from J. Depauw and J-M. Derrien without using any martingale. For the continuous diffusion, the Quenched Central limit theorem for continuous time and discrete space will be proved in detail by a similar way. Finally, we prove the Einstein relation for reversible diffusions in random environment in one dimension.

This thesis is organized as follows:

Chapter 1: We construct again the spectral measure for stationary transformations associated to a function in L^2 and then we give some examples for their applications to the ergodic theorem and the central limit theorem for reversible Markov chain. The proof of the theorem of Kipnis and Varadhan (1986) is showed in detail. We also mention to the spectral measure with values in operator's space.

In chapter 2: We give a new method to obtain the CLT for independence case of variables and for martingale processes.

Chapter 3: The theorems of Gordin and Lifsic for additive functional of stationary Markov chain and then for stationary Markov chain started at a point are reviewed where we use martingale approximation in the proof. These theorems are very classical, but we draw the proofs carefully because they are very useful for the convergence of random walks in random environment in the next chapters.

Chapter 4: This chapter is devoted to CLT for random walks in random environment on $\mathbb{Z}$. In there, the CLT for random walks will be validity if the measurable function c defined on Ω, the space of environments, associated to conductivity of the edge and its inverse belong to L^1. Martingale approximation is used in the proof, adapted from Boivin (1993).

Chapter 5: The main aim of this chapter is to obtain CLT for random walks in random environment in chapter 4 without martingales. More precisely, the convergence is just based on the moments of the variables. An analogue for continuous time and discrete space is given.

Chapter 6: We consider Einstein's relation for Random walk in Random environment by the same method as in the preceding chapter. Assume that there are a drift $\lambda \neq 0$, we will study the convergence of the expectation of Random walk when the drift λ goes to zero.

Chapter 1

Spectral measure for stationary transformations. Applications to Ergodic theorem and Central limit theorem

1.1 Spectral measure for invertible transformation

1.1.1 Invertible stationary transformation

Consider an invertible stationary transformation θ defined on a probability space $(\Omega, \mathcal{A}, \mu)$, such that θ^{-1} is stationary (i.e measure preserving). The associated operator is defined by $Tf = f \circ \theta$. It is an unitary operator if

$$\int_\Omega Tf \cdot \bar{g}\, d\mu = \int_\Omega f \cdot \overline{T^{-1}g}\, d\mu$$

for any $f, g \in L^2(\Omega, \mathbb{C})$.

In the sequel, we will consider T as an operator defined on a stable closed subspace $\mathcal{H} \subset L^2$. An example is $\mathcal{H} = L^2_0$ the space of nul expectation functions.

1.1.2 Spectral measure associated to a function

Let $f \in L^2(\mu)$. We denote by $\mathcal{H}(T, f)$ the smallest Hilbert space which contains all functions $T^k f$, for $k \in \mathbb{Z}$:

$$\mathcal{H}(T, f) = \overline{\left\{ \sum_{k=-n}^{n} a_k T^k f;\ n \geq 1,\ a_{-n}, \dots, a_n \in \mathbb{C} \right\}}^{L_2(\mu)}.$$

Theorem 1.1.1. *Assume $f \in L^2(\mu)$. There exists a positive measure μ_f on $\mathbb{C}$ such that the map Ψ defined on $\mathbb{C}[X]$ by $\Psi\left(\sum_{k=-n}^{n} a_k X^k\right) = \sum_{k=-n}^{n} a_k T^k f$ can be extended to an* *isometry*

$$\begin{array}{rccc} \Psi: & L^2(\mu_f) & \longrightarrow & \mathcal{H}(T,f) \\ & h & \longmapsto & \Psi(h). \end{array}$$

Moreover μ_f *can be chosen such that the operator* Π *defined on* $L^2(\mu_f)$ *by* $(\Pi h)(t) = th(t)$ *satisfies* $\Psi \circ \Pi = T \circ \Psi$.

Proof. For k, ℓ, m integers, we consider

$$\begin{aligned} c(k,\ell) &= \int_\Omega T^k f \cdot \overline{T^\ell f}\, d\mu; \\ \gamma(m) &= \int_\Omega T^m f \cdot \bar{f}\, d\mu. \end{aligned}$$

One has

$$\gamma(k-\ell) = \left\langle T^{k-\ell} f, f \right\rangle_{L^2(\Omega,\mathbb{C})} = \left\langle T^k f, T^\ell f \right\rangle_{L^2(\Omega,\mathbb{C})} = c(k,\ell)$$

and

$$\gamma(k) = \langle T^k f, f\rangle_{L^2(\Omega,\mathbb{C})} = \langle f, T^{-k} f\rangle_{L^2(\Omega,\mathbb{C})} = \overline{\langle T^{-k} f, f\rangle}_{L^2(\Omega,\mathbb{C})} = \overline{\gamma(-k)}.$$

Let $(a_k)_{k=1,\dots,n}$ a finite sequence of complex numbers. Put $g = \sum_{i=0}^n a_i T^i f$ then

$$\begin{aligned} \sum_{i=0}^n \sum_{j=0}^n a_i \overline{a_j} \gamma(i-j) &= \sum_{i=0}^n \sum_{j=0}^n a_i \overline{a_j} \left\langle T^i f, T^j f\right\rangle_{L^2(\Omega,\mathbb{C})} = \left\langle \sum_{i=1}^n a_i T^i f, \sum_{i=0}^n a_i T^i f \right\rangle_{L^2(\Omega,\mathbb{C})} \\ &= \langle g, g\rangle = \|g\|^2_{L^2(\Omega,\mathbb{C})} \geq 0. \end{aligned}$$

Thus, γ is a positive definite function. By the classical Herglotz's theorem, there exists a positive measure μ_f on $[0, 2\pi]$ such that

$$\gamma(k) = \int_0^{2\pi} e^{ik\theta} d\mu_f(\theta)$$

for any positive integer k.

For k is negative integer,

$$\gamma(k) = \overline{\gamma(-k)} = \overline{\int_0^{2\pi} e^{-ik\theta} d\mu_f(\theta)} = \int_0^{2\pi} e^{ik\theta} d\mu_f(\theta).$$

We have thus proved that

$$\gamma(k) = \int_0^{2\pi} e^{ik\theta} d\mu_f(\theta) \tag{1.1}$$

for any k is integer. One also deduces

$$\gamma(0) = \int_0^{2\pi} d\mu_f = \|f\|^2_{L^2(\mu_f)}. \tag{1.2}$$

In the sequel, using the change of variable $\theta \mapsto z = e^{i\theta}$, we consider that μ_f is a measure on $\mathbb{C}$ (with support $\subset S^1 = \{z \in \mathbb{C}, |z| = 1\}$). Thus, formular (1.1) is rewritten as follows

$$\gamma(k) = \int_{S^1} z^k d\mu_f(z). \tag{1.3}$$

Denote $Q[X]$ be the set of polynomials Q such that $Q(X) = \sum_{k=-m}^{m} a_k X^k$. For any polynomial $Q \in Q[X]$, we define

$$\Psi(Q) = \sum_{k=-m}^{m} a_k T^k f. \tag{1.4}$$

For any polynomials $Q_1, Q_2 \in Q[X]$, one has

$$\begin{aligned}
\int_{S^1} Q_1 \overline{Q_2} d\mu_f &= \int_{S^1} \sum_{k=-m_1}^{m_1} a_k z^k \sum_{\ell=-m_2}^{m_2} \overline{b_\ell z^\ell} d\mu_f = \int_{S^1} \sum_{k=-m_1}^{m_1} \sum_{\ell=-m_2}^{m_2} a_k z^k \overline{b_\ell z^\ell} d\mu_f \\
&= \sum_{k=-m_1}^{m_1} \sum_{\ell=-m_2}^{m_2} a_k \overline{b_\ell} \int_{S^1} z^k \overline{z^\ell} d\mu_f = \sum_{k=-m_1}^{m_1} \sum_{\ell=-m_2}^{m_2} a_k \overline{b_\ell} \int_{S^1} z^{k-\ell} d\mu_f \\
&= \sum_{k=-m_1}^{m_1} \sum_{\ell=-m_2}^{m_2} a_k \overline{b_\ell} \gamma(k-\ell) = \sum_{k=-m_1}^{m_1} \sum_{\ell=-m_2}^{m_2} a_k \overline{b_\ell} \langle T^k f, T^\ell f \rangle \\
&= \int_{\Omega} \Psi(Q_1) \overline{\Psi(Q_2)} d\mu.
\end{aligned}$$

It follows that

$$\|\Psi(Q_n)\|_{L^2(\mu)} = \|Q_n\|_{L^2(\mu_f)}. \tag{1.5}$$

Since μ_f has support in $[0, 2\pi]$, for any $h \in L^2(\mu_f)$ then there exists $(Q_n)_{n\geq 1} \subset L^2(\mu_f)$ such that $Q_n \to h$ in L^2. Therefore, for any $\varepsilon > 0$, there exists $M > 0$ such that $\forall n > M$

$$\int_{\mathbb{R}} |Q_n - h|^2 \; d\mu_f < \varepsilon. \tag{1.6}$$

One has $\|\Psi(Q_m) - \Psi(Q_n)\|_{L^2(\mu)} = \|Q_m - Q_n\|_{L^2(\mu_f)} \to 0$ as $m, n \to \infty$. Thus, $\Psi(Q_n)$ is also a Cauchy sequence. Since $L^2(\mu_f)$ is complete, $\Psi(Q_n)$ converges in $L^2(\mu)$ and we denote

$$\Psi(h) = \lim_{n\to\infty} \Psi(Q_n). \tag{1.7}$$

We will show that this limit does not depend on the sequence $(Q_n)_{n\geq 1}$ by the following lemma:

Lemma 1.1.1. *For any sequence $(Q'_n)_{n\geq 1} \to h$ in $L^2(\mu_f)$, then $(\Psi(Q'_n))_{n\geq 1} \to \Psi(h)$ in $L^2(\mu)$.*

Proof. One has

$$\begin{aligned}
\|\Psi(Q'_n) - \Psi(h)\|_{L^2(\mu)} &= \|\Psi(Q'_n) - \Psi(Q_n) + \Psi(Q_n) - \Psi(h)\|_{L^2(\mu)} \\
&\leq \|\Psi(Q'_n) - \Psi(Q_n)\|_{L^2(\mu)} + \|\Psi(Q_n) - \Psi(h)\|_{L^2(\mu)} \\
&\leq \|Q'_n - Q_n\|_{L^2(\mu_f)} + \|\Psi(Q_n) - \Psi(h)\|_{L^2(\mu)} \\
&\leq \|Q'_n - h\|_{L^2(\mu_f)} + \|h - Q_n\|_{L^2(\mu_f)} + \|\Psi(Q_n) - \Psi(h)\|_{L^2(\mu)}
\end{aligned}$$

then (1.6) and (1.7) ensure that $\lim_{n\to\infty} \Psi(Q'_n) = \Psi(h)$. □

By lemma 1.1.1 and by the linearity and continuity of Ψ,

$$\|\Psi(h)\|^2_{\mathcal{H}(T,f)} = \lim_{n\to\infty} \|\Psi(Q_n)\|^2_{\mathcal{H}(T,f)} = \lim_{n\to\infty} \|Q_n\|^2_{L^2(\mu_f)} = \|h\|^2_{L^2(\mu_f)}.$$

We deduce that the map $\Psi : Q \mapsto Q(T)f$ can be extented to a isometry

$$\begin{array}{rccc}
\Psi : & L^2(\mu_f) & \longrightarrow & \mathcal{H}(T,f) \\
& h & \longmapsto & \Psi(h).
\end{array}$$

which proves the first part of Theorem 1.1.1.

Let Π be the operator defined on $L^2(\mu_f)$ by $(\Pi h)(z) = zh(z)$. We will show that $\Psi \circ \Pi = T \circ \Psi$.

⊕ For any polynomial $h(z) = \sum\limits_{k=0}^{n} a_k z^k$, then $\Pi h(z) = \sum\limits_{k=0}^{n} a_k z^{k+1}$. It follows that

$$(\Psi\Pi)h(z) = \sum_{k=0}^{n} a_k T^{k+1} f = T\left(\sum_{k=0}^{n} a_k T^k f\right) = (T\Psi)h(z).$$

⊕ For any $h \in L^2(\mu_f)$. There exists a polynomial h_n which converges to $h \in L^2(\mu_f)$. We have

$$\lim_{i\to\infty} \Pi h_i(z) = \lim_{i\to\infty} z h_i(z) = zh(z)$$

and

$$\Psi(\Pi h_i(z)) = \sum_{k=0}^{n} a_k^{(i)} T^{k+1} f = T\Psi(h_i(z)).$$

Therefore, for $i \to \infty$ we obtain $\Psi(\Pi h(z)) = T\Psi(h(z))$. Hence, we have the result

$$\Psi \circ \Pi = T \circ \Psi. \tag{1.8}$$

□

1.1.3 Application to ergodic theroem

Definition 1.1.1. *The operator T is ergodic if $Th = h$ for some $h \in L^2(\mu)$ then h is constant.*

Theorem 1.1.2. ***(Von Neumann).*** *Assume that T is ergodic. For any $f \in L^2(\mu)$ the following limit holds in L^2:*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k f = \int f \, d\mu. \tag{1.9}$$

Proof. We begin with the following lemma:

Lemma 1.1.2. *For any $z \in \mathbb{C}$ such that $|z| = 1$, then*

$$\lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=0}^{n-1} z^k - \mathbb{1}_{\{1\}}(z) \right)^2 = 0. \tag{1.10}$$

Proof. It is obvious to see that (1.10) holds for $z \in \{-1, 1\}$.

For any $z \in \mathbb{C}/\mathbb{R}$ such that $|z| = 1$, we have

$$\frac{1}{n} \sum_{k=0}^{n-1} z^k - \mathbb{1}_{\{1\}}(z) = \frac{1}{n} \sum_{k=0}^{n-1} z^k = \frac{1}{n} \frac{1 - z^n}{1 - z}$$

which completes

$$\lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=0}^{n-1} z^k - \mathbb{1}_{\{1\}}(z) \right)^2 = 0$$

on $S^1 = \{z \in \mathbb{C}, |z| = 1\}$. □

Proof of theorem 1.1.2. Since $\left| \frac{1}{n} \sum\limits_{k=0}^{n-1} z^k - \mathbb{1}_{\{1\}}(z) \right| \leq 2$, the dominated convergence theorem ensures that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \int \left| \frac{1}{n} \sum_{k=0}^{n-1} z^k - \mathbb{1}_{\{1\}}(z) \right|^2 d\mu_f = \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} z^k - \mathbb{1}_{\{1\}}(z) \right\|_{L^2(\mu_f)} \\ &= \lim_{n\to\infty} \left\| \Psi \left(\frac{1}{n} \sum_{k=0}^{n-1} z^k - \mathbb{1}_{\{1\}}(z) \right) \right\|_{\mathcal{H}(T,f)}. \end{aligned}$$

We have thus proved

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T^k f = h \text{ in } L^2 \text{ with } h = \Psi\left(\mathbb{1}_{\{1\}}(z)\right). \tag{1.11}$$

Moreover, since $z\mathbb{1}_{\{1\}}(z) = \mathbb{1}_{\{1\}}(z), \quad \forall z \in \mathbb{C}$ implies that $\Psi\left(z\mathbb{1}_{\{1\}}(z)\right) = \Psi\left(\mathbb{1}_{\{1\}}(z)\right)$. Using the fact $\Psi \circ \Pi(h) = T \circ \Psi(h)$, one has $\Psi\left(z\mathbb{1}_{\{1\}}(z)\right) = T \circ \Psi\left(\mathbb{1}_{\{1\}}(z)\right)$ and hence $Th = h$. It follows that $h = c$ (constant) since T is ergodic. And since the transformation is stationary,

$$\int T^k f d\mu = \int f d\mu, \quad \forall k \geq 0$$

and so

$$\int \frac{1}{n} \sum_{k=0}^{n-1} T^k f \, d\mu = \int f \, d\mu. \tag{1.12}$$

Combine (1.11) and (1.12) one has

$$\lim_{n\to\infty} \int \frac{1}{n} \sum_{k=0}^{n-1} T^k f \, d\mu = c = \int f \, d\mu.$$

which completes the proof of theorem 1.1.2. □

1.2 Spectral measure for reversible Markov chain

1.2.1 Markov Chain

Suppose $(X_n)_{n\geq 0}$ is a stationary Markov chain defined on a probability space $(\Omega, \mathcal{A}, \mu)$ with μ-initial distribution and $(\mathcal{X}, \mathcal{B})$ be the state space. A stochastic kernel (transtion probability) is a map $P : \mathcal{X} \times \mathcal{B} \to [0; 1]$ such that:

- $x \longmapsto P(x, A)$ is $\mathcal{B}$-measurable for any $A \subset \mathcal{B}$.
- $A \longmapsto P(x, A)$ is a probability measure for any $x \in \mathcal{X}$.

It also acts on the space $B(\mathcal{X})$ of bounded, measurable functions by

$$Pf(x) = \mathbb{E}\left\{f(X_1)/X_0 = x\right\}. \tag{1.13}$$

1.2.2 Reversible Markov Chain

Consider a Markov operator P defined on a probability space $(\Omega, \mathcal{A}, \mu)$. We suppose that the associated Markov chain $(X_n)_{n\geq 0}$ with initial law μ is reversible, i.e.:

Definition 1.2.1. *The Markov chain $(X_n)_{n\geq 0}$ with transition operator P and initial law μ is reversible is $P = P^\star$ in $L^2(\mu)$:*

$$\int_\Omega Pf \cdot \bar{g}\ d\mu = \int_\Omega f \cdot \overline{Pg}\ d\mu$$

for any $f, g \in L^2(\Omega, \mathbb{C})$.

In this situation, $(X_n)_{n\geq 0}$ is a stationary Markov chain, i.e $\int Pf\ d\mu = \int f\ d\mu$.

In the sequel, we will consider P as an operator defined on a stable closed subspace $\mathcal{H} \subset L^2$. We recall

$$\|P\|_{\mathcal{H}} = \sup_{\|f\|\neq 0} \frac{\|Pf\|_{L^2(\mu)}}{\|f\|_{L^2(\mu)}} \tag{1.14}$$

so we have $\|P\|_{\mathcal{H}} \leq 1$ (but not necessary $= 1$). An example is $\mathcal{H} = L^2_0$ the space of nul expectation functions.

1.2.3 Spectral measure associated to a function

Let $f \in L^2(\mu)$. We denote by $\mathcal{H}(P, f)$ the smallest Hilbert space which contains all functions $P^k f$, for $k \geq 0$:

$$\mathcal{H}(P, f) = \overline{\left\{\sum_{k=0}^{n} a_k P^k f;\ n \geq 0,\ a_k \in \mathbb{C}\right\}}^{L_2(\mu)}.$$

Theorem 1.2.1. *Assume $f \in L^2(\mu)$. There exists a positive measure μ_f on $\mathbb{R}$ such that the map Ψ defined on $\mathbb{C}[X]$ by $\Psi\left(\sum_{k=0}^n a_k X^k\right) = \sum_{k=0}^n a_k P^k f$ can be extended to an isometry*

$$\begin{array}{rccc} \Psi: & L^2(\mu_f) & \longrightarrow & \mathcal{H}(P,f) \\ & h & \longmapsto & \Psi(h). \end{array}$$

Moreover μ_f can be chosen such that the operator Π defined on $L^2(\mu_f)$ by $(\Pi h)(t) = th(t)$ satisfies $\Psi \circ \Pi = P \circ \Psi$.

Proof. For k, ℓ, m positive integers, we consider

$$\begin{aligned} c(k,\ell) &= \int_\Omega P^k f \cdot \overline{P^\ell f}\, d\mu; \\ \gamma(m) &= \int_\Omega P^m f \cdot \bar{f}\, d\mu. \end{aligned}$$

and for $s, t, u \in \mathbb{R}$

$$\begin{aligned} \psi(s,t) &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{(is)^k(-it)^\ell}{k!\ell!} c(k,\ell); \\ \phi(u) &= \sum_{m=0}^{\infty} \frac{(iu)^m}{m!}\gamma(m). \end{aligned}$$

One has

$$\begin{aligned} \psi(s,t) &= \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{(is)^k(-it)^\ell}{k!\ell!} \left\langle P^k f, P^l f\right\rangle = \left\langle e^{isP} f, e^{itP} f\right\rangle \\ &= \left\langle e^{i(s-t)P} f, f\right\rangle = \phi(s-t) \end{aligned}$$

since $\overline{e^{itP}} = e^{-itP}$ and $P = P^*$. Hence, $\psi(s,t) = \phi(s-t)$.

Moreover $|\phi(u)| = |\langle e^{iuP} f, f\rangle| \leq \|e^{iuP} f\|_{L^2(\mu)} \|f\|_{L^2(\mu)} \leq \|f\|^2_{L^2(\mu)}$. Then, the dominated convergence theorem follows that $\lim\limits_{u\to 0} \phi(u) = \|f\|^2_{L^2(\mu)}$. In addition, $\phi(0) = \|f\|^2_{L^2(\mu)}$, follows that ϕ is continuous at 0.

Let $(a_k)_{k=1,\dots,n}$ a finite sequence of complex numbers, and $(s_k)_{k=1,\dots,n}$ a finite sequence of real numbers.

$$\sum_{k=1}^{n}\sum_{\ell=1}^{n} a_k \bar{a}_\ell \phi(s_k - s_\ell) = \left\langle \sum_{k=1}^{n} a_k \sum_{m=0}^{\infty} \frac{(is_k)^m}{m!} P^m f, \sum_{k=1}^{n} a_k \sum_{m=0}^{\infty} \frac{(is_k)^m}{m!} P^m f \right\rangle.$$

Put $g = \sum_{k=1}^n a_k \sum_{m=0}^\infty \frac{(is_k)^m}{m!} P^m f$, one has

$$\begin{aligned} \|g\| &\leq \sum_{k=1}^{n} \left\| a_k \sum_{m=0}^{\infty} \frac{(is_k)^m}{m!} P^m f \right\| \leq \sum_{k=1}^{n} |a_k| \cdot \left\| \sum_{m=0}^{\infty} \frac{(is_k)^m}{m!} P^m f \right\| \\ &\leq \sum_{k=1}^{n} |a_k| \cdot \sum_{m=0}^{\infty} \frac{|is_k|^m}{m!} \|f\| \leq \sum_{k=1}^{n} |a_k| \cdot e^{s_k^2} \|f\| < \infty. \end{aligned}$$

hence, $g \in L^2(\mu)$ and $\sum_{k=1}^n \sum_{\ell=1}^n a_k \bar{a}_\ell \phi(s_k - s_\ell) = \|g\|^2_{L^2(\mu)} \geq 0$.

Thus, ϕ is a positive definite function. By the classical Bochner's theorem, there exists a positive measure μ_f on $\mathbb{R}$ such that

$$\phi(u) = \int_{\mathbb{R}} e^{iut} d\mu_f(t) = \hat{\mu}_f(u). \tag{1.15}$$

One also deduces that

$$\hat{\mu}_f(0) = \int_{\mathbb{R}} d\mu_f = \phi(0) = \|f\|^2_{L^2(\mu)}. \tag{1.16}$$

By the definition of derivative of ϕ

$$\begin{aligned} \phi'(u) &= \lim_{h\to 0} \frac{\phi(u+h) - \phi(u)}{h} = \lim_{h\to 0} \int \frac{(e^{i(u+h)t} - e^{iut})}{h} d\mu_f \\ &= \lim_{h\to 0} \int e^{iut} \frac{(e^{iht} - 1)}{h} d\mu_f \end{aligned}$$

since $\left| \frac{(e^{iht}-1)}{h} \right| \leq 2|t| < \infty$, the dominated convergence theorem follows

$$\phi'(u) = i \int t e^{iut} d\mu_f$$

and similarly

$$\phi^m(u) = i^m \int t^m e^{iut} d\mu_f.$$

Furthermore, by computing directly the derivatives of ϕ, we also have

$$\phi^m(0) = i^m \gamma(m)$$

Hence, one has

$$\gamma(m) = \langle P^m f, f \rangle = \int t^m d\mu_f. \tag{1.17}$$

Denote $Q\,[X]$ be the set of polynomials Q such that $Q(X) = \sum\limits_{k=0}^{m} a_k X^k$. For any polynomial $Q \in Q\,[X]$, we define

$$\Psi(Q) = \sum_{k=0}^{m} a_k P^k f.$$

Then, for any polynomials $Q_1, Q_2 \in Q\,[X]$ we have

$$\begin{aligned} \int_{\mathbb{R}} Q_1 \overline{Q_2} d\mu_f &= \sum_{k=0}^{m_1} \sum_{\ell=0}^{m_2} a_k \overline{b_\ell} \int_{\mathbb{R}} t^{k+\ell} d\mu_f = \sum_{k=0}^{m_1} \sum_{\ell=0}^{m_2} a_k \overline{b_\ell} \gamma(k+\ell) \\ &= \sum_{k=0}^{m_1} \sum_{\ell=0}^{m_2} a_k \overline{b_\ell} c(k,\ell) = \int_{\Omega} \Psi(Q_1) \overline{\Psi(Q_2)} d\mu. \end{aligned}$$

It follows that

$$\langle \Psi(Q_1), \Psi(Q_2) \rangle_{L^2(\mu)} = \langle Q_1, Q_2 \rangle_{L^2(\mu_f)}$$

and hence

$$\|\Psi(Q_n)\|_{L^2(\mu)} = \|Q_n\|_{L^2(\mu_f)}. \tag{1.18}$$

Lemma 1.2.1. *μ_f has a bounded support.*

Proof. For any $g \in \mathcal{H}(P, f)$, then $g = \sum_{k=0}^{n} a_k P^k f$ for some $a_k \in \mathbb{C}$.

Put $Q(t) = \sum_{k=0}^{n} a_k t^k$. We have

$$\|P(g)\|^2_{L^2(\mu_f)} = \|P(\Psi(Q))\|^2_{L^2(\mu_f)} \leq \|P\|^2_{\mathcal{H}(P,f)} \|\Psi(Q)\|^2_{L^2(\mu_f)}$$

then

$$\int t^2 |Q(t)|^2 d\mu_f \leq \|P\|^2_{\mathcal{H}(P,f)} \int |Q(t)|^2 d\mu_f.$$

It follows that $|t| \leq \|P\|_{\mathcal{H}(P,f)}, \mu_f$ *a.s.* So, support of $\mu_f \subset \left[-\|P\|_{\mathcal{H}(P,f)}, \|P\|_{\mathcal{H}(P,f)}\right]$. ∎

By lemma 1.2.1, for any $h \in L^2(\mu_f)$ then there exists $(Q_n)_{n\geq 1} \subset L^2(\mu_f)$ such that $Q_n \to h$ in L^2. So, for any $\varepsilon > 0$, there exists $M > 0$ such that for any $n > M$ then

$$\int_{\mathbb{R}} |Q_n - h|^2 \, d\mu_f < \varepsilon. \tag{1.19}$$

Furthermore, $(Q_n)_{n\geq 1}$ is also a Cauchy sequence, and so we have

$$\|\Psi(Q_m) - \Psi(Q_n)\|_{L^2(\mu)} = \|Q_m - Q_n\|_{L^2(\mu_f)} \to 0 \;\; as \;\; m, n \to \infty$$

Hence $\Psi(Q_n)$ is a Cauchy sequence also. Since $L^2(\mu_f)$ is complete, $\Psi(Q_n)$ converges in $L^2(\mu)$ and denote

$$\Psi(h) = \lim_{n\to\infty} \Psi(Q_n). \tag{1.20}$$

Lemma 1.2.2. *For any sequence $(Q'_n)_{n\geq 1} \to h$ in $L^2(\mu_f)$, then $(\Psi(Q'_n))_{n\geq 1} \to \Psi(h)$ in $L^2(\mu)$.*

Proof. One has

$$\begin{array}{rcl} \|\Psi(Q'_n) - \Psi(h)\| & = & \|\Psi(Q'_n) - \Psi(Q_n) + \Psi(Q_n) - \Psi(h)\| \\ & \leq & \|\Psi(Q'_n) - \Psi(Q_n)\| + \|\Psi(Q_n) - \Psi(h)\| \\ & \leq & \|Q'_n - Q_n\| + \|\Psi(Q_n) - \Psi(h)\| \\ & \leq & \|Q'_n - h\| + \|h - Q_n\| + \|\Psi(Q_n) - \Psi(h)\| \end{array}$$

then (1.19) and (1.20) ensure that $\lim_{n\to\infty} \|\Psi(Q'_n) - \Psi(h)\| = 0$. ∎

Therefore, by the linearity and continuity of Ψ,

$$\|\Psi(h)\|^2_{\mathcal{H}(P,f)} = \lim_{n\to\infty} \|\Psi(Q_n)\|^2_{\mathcal{H}(P,f)} = \lim_{n\to\infty} \|Q_n\|^2_{L^2(\mu_f)} = \|h\|^2_{L^2(\mu_f)}.$$

We deduce that the map $\Psi : Q \mapsto Q(P)f$ can be extented to a isometry

$$\begin{array}{cccc} \Psi : & L^2(\mu_f) & \longrightarrow & \mathcal{H}(P,f) \\ & h & \longmapsto & \Psi(h). \end{array}$$

which proves the first part of Theorem 1.2.1.

Let Π the operator defined on $L^2(\mu_f)$ by $(\Pi h)(t) = th(t)$. For any polynomial $h(t) = \sum_{k=0}^{n} a_k t^k$. We have $\Pi h(t) = \sum_{k=0}^{n} a_k t^{k+1}$ and then

$$(\Psi\Pi)h(t) = \sum_{k=0}^{n} a_k P^{k+1} f = P\left(\sum_{k=0}^{n} a_k P^k f\right) = (P\Psi)(h(t)).$$

For any $h \in L^2(\mu_f)$. There exists a polynomial h_n which converges to $h \in L^2(\mu_f)$. We have

$$\lim_{i\to\infty} \Pi h_i(t) = \lim_{i\to\infty} th_i(t) = th(t)$$

and

$$\Psi(\Pi h_i(t)) = \sum_{k=0}^{n} a_k^{(i)} P^{k+1} f = P\Psi(h_i(t)).$$

Therefore, for $i \to \infty$ we obtain $\Psi(\Pi h(t)) = P\Psi(h(t))$. Hence, we have

$$\Psi \circ \Pi = P \circ \Psi \tag{1.21}$$

which completes the proof of Theorem 1.2.1. □

Denote $S(\mu_f)$ the support of μ_f:

$$S(\mu_f) = \{t : \forall\varepsilon > 0, \mu_f\,[t-\varepsilon, t+\varepsilon] > 0\}.$$

Proposition 1.2.1. *We have* $\|P\|_{\mathcal{H}(P,f)} = \sup_{t\in S(\mu_f)} |t|$.

Proof. Since Ψ is an isometry from $L^2(\mu_f)$ onto $\mathcal{H}(P,f)$

$$\begin{aligned} \|P\|_{\mathcal{H}(P,f)} &= \sup_{\|g\|_{\mathcal{H}(P,f)}=1} \|P(g)\|_{\mathcal{H}(P,f)} = \sup_{\|\Psi(h)\|_{\mathcal{H}(P,f)}=1} \|P\cdot\Psi(h)\|_{\mathcal{H}(P,f)} \\ &= \sup_{\|h\|_{L^2(\mu_f)}=1} \|\Psi\cdot\Pi(h)\|_{\mathcal{H}(P,f)} = \sup_{\|h\|_{L^2(\mu_f)}=1} \|\Pi(h)\|_{L^2(\mu_f)} \\ &= \sup_{\|h\|_{L^2(\mu_f)}=1} \|th(t)\|_{L^2(\mu_f)}. \end{aligned}$$

We have also

$$\sup_{\|h\|_{L^2(\mu_f)}=1} \|th(t)\|_{L^2(\mu_f)} \le \sup_{\substack{\|h\|_{L^2(\mu_f)}=1 \\ t\in S(\mu_f)}} |t|\cdot\|h(t)\|_{L^2(\mu_f)} \le \sup_{t\in S(\mu_f)} |t|.$$

We will prove that this inequalities is equalities. Put $t_0 = \sup_{t\in S(\mu_f)} |t|$ and for each $1 \leq n \in \mathbb{N}$, let $t_n \in S(\mu_f)$ such that $|t_n - t_0| < \dfrac{1}{n}$.

Choose $h_n = \dfrac{1}{\sqrt{c_n}}\mathbf{1}_{B(t_n,1/n)}$ where $c_n = \displaystyle\int_{t_n-\frac{1}{n}}^{t_n+\frac{1}{n}} d\mu_f > 0$ since $t_n \in S(\mu_f)$ and $B(t_n, 1/n)$ be the open balls have radius $1/n$ and center at t_n, then $\|h_n\|_{L^2(\mu_f)} = 1$. By computing,

$$\|th_n\|_{L^2(\mu_f)} = \frac{1}{\sqrt{c_n}}\sqrt{\int_{t_n-1/n}^{t_n+1/n} t^2 d\mu_f}, \quad \forall n \geq 1.$$

For n is large enough,

$$\|th_n\|_{L^2(\mu_f)} \approx |t_n| \approx t_0$$

Hence,

$$\sup_{\|h_n\|_{L^2(\mu_f)}} \|th_n\|_{L^2(\mu_f)} = t_0 = \sup_{t\in S(\mu_f)} |t|$$

so,

$$\|P\|_{\mathcal{H}(P,f)} = \sup_{\|h\|_{L^2(\mu_f)}} \|th\|_{L^2(\mu_f)} = \sup_{t\in S(\mu_f)} |t|.$$

□

Corollary 1.2.1. $S(\mu_f) \subset [-1, 1]$.

Indeed, since $\|P\|_{\mathcal{H}(P,f)} = \sup_{t\in S(\mu_f)} |t| \leq 1$ we obtain the desired result.

1.2.4 Application to ergodic theorem

Definition 1.2.2. *P is ergodic if $Ph = h$ for some $h \in L^2(\mu)$ then h is constant.*

Proposition 1.2.2. *Assume that P is ergodic. For any $f \in L^2(\mu)$ the following limit holds in L^2:*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} P^k f = \int f \, d\mu. \tag{1.22}$$

Proof. Consider

$$\frac{1}{n}\sum_{k=0}^{n-1} t^k \longrightarrow \begin{cases} 0 & if \quad |t| < 1 \\ 0 & if \quad t = -1 \\ 1 & if \quad t = 1 \end{cases}$$

Then we have

$$\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=0}^{n-1} t^k - \mathbb{1}_{\{1\}}(t)\right)^2 = 0 \text{ on } [-1,1]\,.$$

Since $\left|\frac{1}{n}\sum_{k=0}^{n-1} t^k - \mathbb{1}_{\{1\}}(t)\right| \leq 2$, the dominated convergence theorem ensures that

$$\lim_{n\to\infty} \int \left|\frac{1}{n}\sum_{k=0}^{n-1} t^k - \mathbb{1}_{\{1\}}(t)\right|^2 d\mu_f = 0$$

so

$$\lim_{n\to\infty} \left\|\Psi\left(\frac{1}{n}\sum_{k=0}^{n-1} t^k - \mathbb{1}_{\{1\}}(t)\right)\right\|_{\mathcal{H}(P,f)} = \lim_{n\to\infty} \left\|\frac{1}{n}\sum_{k=0}^{n-1} t^k - \mathbb{1}_{\{1\}}(t)\right\|_{L^2(\mu_f)} = 0.$$

It follows that

$$\lim_{n\to\infty} \left\|\frac{1}{n}\sum_{k=0}^{n-1} P^k f - h\right\| = 0 \text{ in } L^2 \text{ with } h = \Psi(\mathbb{1}_{\{1\}}(t))$$

and hence, we obtain

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} P^k f = h \text{ in } L^2.$$

Moreover,

$$t\mathbb{1}_{\{1\}}(t) = \mathbb{1}_{\{1\}}(t), \quad \forall t \in \mathbb{R}$$

so

$$\Psi(t\mathbb{1}_{\{1\}}(t)) = \Psi(\mathbb{1}_{\{1\}}(t)) \Longrightarrow Ph = h \Longrightarrow h = c$$

where c is a constant since P is ergodic. On the other hand, since the Markov chain is stationary,

$$\int P^k f d\mu = \int f d\mu, \quad \forall k \geq 0$$

then

$$\int \frac{1}{n}\sum_{k=0}^{n-1} P^k f \, d\mu = \int f \, d\mu \Longrightarrow c = \int f \, d\mu.$$

Hence, we deduce that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} P^k f = \int f \, d\mu \text{ in } L^2.$$

□

1.2.5 Application to Central limit theorem

1.2.5.1 Variance principle

Proposition 1.2.3. *Assume that $f \in L_0^2(\mu)$. There exists $g \in \mathcal{H}(P,f)$ such that $f = g - Pg$ if and only if*

$$\int_{-1}^{1} \frac{1}{(1-t)^2} \, d\mu_f(t) < +\infty. \tag{1.23}$$

In this case, setting $\sigma_f^2 = \|g\|^2 - \|Pg\|^2$ *we have*

$$\sigma_f^2 = \int_{-1}^{1} \frac{1+t}{1-t}\, d\mu_f(t). \tag{1.24}$$

Proof. We will prove the sufficient and necessary conditions of this lemma.

Suppose that (1.23) holds , then $h(t) = \dfrac{1}{1-t} \in L^2(\mu_f)$, and hence $(1-t)h(t) = 1 \in L^2(\mu_f)$. It follows that $\Psi(h) - \Psi(th) = \Psi(1) = f$. Put $g = \Psi(h)$, then $f = g - Pg$.

Conversely, if there exists $g \in \mathcal{H}(P, f)$ such that $f = g - Pg$. We recall the operator Ψ which is isometry

$$\begin{array}{rccl} \Psi: & L^2(\mu_f) & \longrightarrow & \mathcal{H}(P,f) \\ & 1 & \longmapsto & f = \Psi(1) \\ & h & \longmapsto & g = \Psi(h). \end{array}$$

One has

$$Pg = P(\Psi(h)) = \Psi(\Pi(h)) = \Psi(th(t)).$$

Since $f = g - Pg$, then $\Psi(1) = \Psi(h(t)) - \Psi(th(t))$ and so $\Psi(1 - h(t) + th(t)) = 0$. It follows that $1 - h(t) + th(t) = 0$, implies $h(t) = \dfrac{1}{1-t} \in L^2(\mu_f)$. Hence, we obtain (1.23).

We deduce also

$$\begin{aligned} \sigma_f^2 &= \|g\|^2_{\mathcal{H}(P,f)} - \|Pg\|^2_{\mathcal{H}(P,f)} = \|\Psi(h)\|^2_{\mathcal{H}(P,f)} - \|\Psi(\Pi(h))\|^2_{\mathcal{H}(P,f)} \\ &= \|h\|^2_{L^2(\mu_f)} - \|\Pi(h)\|^2_{L^2(\mu_f)} = \int_{-1}^{1} \left(\frac{1}{1-t}\right)^2 d\mu_f(t) - \int_{-1}^{1} \left(\frac{t}{1-t}\right)^2 d\mu_f(t) \\ &= \int_{-1}^{1} \frac{1+t}{1-t} d\mu_f(t). \end{aligned}$$

which completes (1.24). □

We consider the power series expansion $(1-t)^{1/2} = 1 - \sum_{j=1}^{\infty} a_j t^j$, where $a_1 = 1/2$ and

$$a_j = \frac{\frac{1}{2}(1-\frac{1}{2})\dots(j-1-\frac{1}{2})}{j!} \quad for\ j \geq 2.$$

We have $a_j > 0$ for $j \geq 1$ and $\sum_{j=1}^{\infty} a_j = 1$, so for a contraction P in a Banach space $L^2(\mu)$ the series $\sum_{j=1}^{\infty} a_j P^j$ is absolutely convergent in the operator norm, and defines a contraction $P_{1/2}$ (see Derriennic and Lin [13], page 95).

Definition 1.2.3. *For a contraction P in a Banach space $L^2(\mu)$, we define*

$$\sqrt{I-P} = I - P_{1/2} = I - \sum_{j=1}^{\infty} a_j P^j.$$

***Remark* 1.2.1.** *There is another definition of $\sqrt{I-P}$ with spectral theory (see (1.56) in remark 1.3.2, section 1.3).*

Proposition 1.2.4. *Assume that $f \in L_0^2(\mu)$. There exists $g' \in \mathcal{H}(P, f)$ such that $f = \sqrt{I-P}g'$ if and only if*

$$\int_{-1}^{1} \frac{1+t}{1-t}\, d\mu_f(t) < +\infty. \tag{1.25}$$

Proof. Suppose (1.25) holds, then $\int_{-1}^{1} \frac{1}{1-t} d\mu_f(t) < \infty$ since $\frac{1+t}{1-t} + 1 = \frac{2}{1-t}$. Put $h(t) = \frac{1}{\sqrt{1-t}} \in L^2(\mu_f)$, then

$$1 = \sqrt{1-t}.h(t) \in L^2(\mu_f). \tag{1.26}$$

Lemma 1.2.3. *Put $\varphi_1 = \sqrt{1-t} \in L^2(\mu_f)$. Assume that φ_2 and $\varphi_1\varphi_2 \in L^2(\mu_f)$ then*

$$\Psi\left(\varphi_1\varphi_2\right) = \varphi_1(P) \circ \Psi\left(\varphi_2\right). \tag{1.27}$$

Proof. There exists a sequence of polynomials $Q_n = 1 - \sum_{j=1}^{n} a_j t^j$ converges to φ_1 in $L^2(\mu_f)$. One has

$$Q_n\varphi_2 = (1 - \sum_{j=1}^{n} a_j t^j)\varphi_2 \in L^2(\mu_f)$$

since Q_n continuous on $[-1, 1]$ and so bounded. Applying formula $\Psi \circ \Pi = P \circ \Psi$, then

$$\begin{aligned} \Psi(Q_n\varphi_2) &= \Psi(\varphi_2) + \Psi(\sum_{j=1}^{n} a_j t^j \varphi_2) = \Psi(\varphi_2) + P \circ \Psi(\sum_{j=1}^{n} a_j t^{k-1}\varphi_2) \\ &= \left[I + \sum_{j=1}^{n} a_j P^j \right] \circ \Psi(\varphi_2) = Q_n(P) \circ \Psi(\varphi_2). \end{aligned}$$

For $n \to \infty$, the bracket tend to $\sqrt{I-P}$, one has $\Psi\left(\varphi_1\varphi_2\right) = \sqrt{I-P} \circ \Psi\left(\varphi_2\right)$. □

Apply lemma 1.2.3 for (1.26), , one has

$$\Psi(1) = \sqrt{I-P}\Psi(h) = f.$$

Put $g' = \Psi(h) \in \mathcal{H}(P, f)$, then $f = \sqrt{I-P}g'$.

Conversely, if there exists $g' \in \mathcal{H}(P, f)$ such that $\sqrt{I-P}g' = f$. Put

$$q_1 = \sqrt{1-t}, \quad q_2 = \Psi^{-1}(g')$$

then q_1, q_2 and $q_1q_2 \in L^2(\mu_f)$. Applying lemma 1.2.3, one has

$$\Psi(q_1q_2) = q_1(P) \circ \Psi(q_2) = f = \Psi(1).$$

It follows that

$$\Psi\left(1 - q_1q_2\right) = 0$$

then

$$q_2 = \frac{1}{q_1} = \frac{1}{\sqrt{1-t}} \in L^2(\mu_f)$$

which completes (1.25). □

We recall here Markov chain $(X_n)_{n\geq 0}$ with initial law μ is reversible. Denote

$$S_n = \sum_{k=1}^{n} f(X_k).$$

Proposition 1.2.5. *Assume that $f \in L^2_0(\mu)$. Then $\sup_n \mathbb{E}\left\{\frac{S_n^2}{n}\right\}$ is finite if and only if (1.25) holds.*

Proof. Firstly, one has

$$\begin{aligned}\mathbb{E}\left\{f^2(X_k)/X_0 = x\right\} &= \int f^2(y)P^k(x,dy) = \int f^2(y)\int P^{k-1}(x,dz)P(z,dy)\\ &= \int Pf^2(z)P^{k-1}(x,dz) = \ldots = \int P^{k-1}f^2(t)P(x,t)\\ &= P^k f^2(x)\end{aligned}$$

and for $1 \leq k < \ell \leq n$,

$$\begin{aligned}\mathbb{E}\left\{f(X_k)f(X_\ell)/X_0\right\} &= \mathbb{E}\left\{\mathbb{E}\left\{f(X_k)f(X_\ell)/X_k\right\}/X_0\right\} = \mathbb{E}\left\{f(X_k)\mathbb{E}\left\{f(X_\ell)/X_k\right\}/X_0\right\}\\ &= \mathbb{E}\left\{f(X_k)P^{\ell-k}f(X_k)/X_0\right\} = P^k(fP^{\ell-k}f)(X_0).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}\left\{\frac{S_n^2}{n}\right\} &= \mathbb{E}\left\{\frac{1}{n}\mathbb{E}\left\{S_n^2/X_0\right\}\right\} = \frac{1}{n}\sum_{k=1}^{n}\mathbb{E}\left\{P^k(f^2)(X_0)\right\} + \frac{2}{n}\sum_{1\leq k<\ell\leq n}\mathbb{E}\left\{P^k(f\cdot P^{\ell-k}f)(X_0)\right\}\\ &= \frac{1}{n}\sum_{k=1}^{n}\int P^k(f^2)d\mu + 2\sum_{1\leq k<\ell\leq n}\int P^k(f\cdot P^{\ell-k}f)\,d\mu\\ &= \int f^2 d\mu + \frac{2}{n}\sum_{1\leq k<\ell\leq n}\left\langle f, P^{\ell-k}f\right\rangle = \int_{-1}^{1}\left(1+\frac{2}{n}\sum_{1\leq k<\ell\leq n}t^{\ell-k}\right)\,d\mu_f(t)\\ &= \int_{-1}^{1} h_n(t)\,d\mu_f(t)\end{aligned}$$

with $h_n(t) = 1 + \frac{2}{n}\sum_{k=1}^{n-1}\sum_{\ell=1}^{k} t^\ell$.

Lemma 1.2.4. *We have*

$$\lim_{n\to\infty} h_n(t) \to \begin{cases}\frac{1+t}{1-t} & if \quad -1<t<1\\ 0 & if \quad t=-1\end{cases}$$

Moreover, if $t\in[0,1)$ then the limit is monotone; and if $t \in [-1,0)$ then $|h_n(t)| \leq 1$.

Proof. • Consider the case $|t| < 1$,

$$\lim_{n\to\infty} h_n(t) = \lim_{n\to\infty}\left(1+\frac{2}{n}\sum_{k=1}^{n-1}\frac{t-t^k}{1-t}\right) = \frac{1+t}{1-t}.$$

- Consider $t = -1$,

$$\begin{aligned} h_{2n}(-1) &= 0, \\ \lim_{n\to\infty} h_{2n+1}(-1) &= \lim_{n\to\infty}\left(-1 + 2\frac{n+1}{2n+1}\right) = 0. \end{aligned}$$

  One obtains

$$\lim_{n\to\infty} h_n(-1) = 0.$$

- It is clear that if $t \in [0,1)$ then $h_n(t)$ is a positive increase sequence, the limit is monotone.

- Consider $t \in [-1,0)$, then $-1 \leq \sum_{\ell=1}^{k} t^{\ell} < 0$. It follows that $-1 + 2/n \leq h_n(t) \leq 1$ implies $|h_n(t)| \leq 1$.

□

Denote by $\mathcal{M}$ the space of invariant functions by P, that is

$$\mathcal{M} = \left\{\varphi \in L^2(\mu) : P\varphi = \varphi\right\}.$$

Lemma 1.2.5. *For any $f \in \mathcal{M}^{\perp}$ in $L^2(\mu)$, then $\mu_f(\{1\}) = 0$.*

Proof. For any $\varphi \in \mathcal{M}$

$$\begin{aligned} 0 &= \langle f, \varphi\rangle = \langle f, P\varphi\rangle = \langle Pf, \varphi\rangle = \left\langle P^k f, \varphi\right\rangle, \ \forall k \geq 0 \\ &= \left\langle \sum_{k=0}^{n} a_k P^k f, \varphi\right\rangle, \ \forall n \geq 0, a_k \in \mathbb{C} \end{aligned}$$

It follows that $\mathcal{M} \perp \mathcal{H}(P, f)$.

Let $h = \mathbb{1}_{\{1\}}(t) \in L^2(\mu_f)$ then, there exists $g = \Psi(h) \in \mathcal{H}(P, f)$ such that

$$\|g\|^2_{\mathcal{H}(P,f)} = \|h\|^2_{L^2(\mu_f)} = \int \mathbb{1}_{\{1\}}(t)\, d\mu_f(t) = \mu_f(\{1\})$$

On the other hand, by the definition of function h we have $th(t) = h(t), \forall t \in \mathbb{R}$, then

$$\begin{aligned} \Psi(th(t)) &= P(\Psi h(t)) = Pg \\ \Psi(th(t)) &= \Psi(h(t)) = g \end{aligned}$$

and so $Pg = g$. It follows that $g \in \mathcal{M}$ implies $g \in \mathcal{M} \cap \mathcal{H}(P, f) = \{0\}$ then $\|g\|^2_{\mathcal{H}(P,f)} = 0$ and hence $\mu_f(\{1\}) = 0$. □

Now, we return the proof of the proposition 1.2.5. **Firstly, we prove the necessary condition**. Assume that $\int_{-1}^{1} \frac{1+t}{1-t} d\mu_f(t) = +\infty$, we have

$$+\infty = \int_{-1}^{1} \frac{1+t}{1-t} d\mu_f(t) = \int_{-1}^{0} \frac{1+t}{1-t} d\mu_f(t) + \int_{0}^{1} \frac{1+t}{1-t} d\mu_f(t)$$

$$\leq \int_{-1}^{1} d\mu_f(t) + \int_{0}^{1} \frac{1+t}{1-t} d\mu_f(t)$$

then we obtain $\int_0^1 \frac{1+t}{1-t} d\mu_f(t) = +\infty$. By the monotone convergence theorem,

$$\lim_{n\to\infty} \int_0^1 h_n(t) d\mu_f(t) = \int_0^1 \lim_{n\to\infty} h_n(t) d\mu_f(t) = \int_0^1 \frac{1+t}{1-t} d\mu_f(t) = +\infty.$$

Moreover, by the dominated convergence theorem, one has

$$0 \leq \lim_{n\to\infty} \int_{-1}^0 h_n(t) d\mu_f(t) = \int_{-1}^0 \lim_{n\to\infty} h_n(t) d\mu_f(t) = \int_{-1}^0 \frac{1+t}{1-t} d\mu_f(t) < +\infty.$$

We have thus proved that

$$\lim_{n\to\infty} \int_{-1}^1 h_n(t) d\mu_f(t) = \lim_{n\to\infty} \int_{-1}^0 h_n(t) d\mu_f(t) + \lim_{n\to\infty} \int_0^1 h_n(t) d\mu_f(t) = +\infty$$

Thus, $\lim_{n\to\infty} \mathbb{E}\left\{S_n^2/n\right\} = +\infty$. This is a contradiction.

Conversely, if (1.25) holds then

$$\begin{aligned} \mathbb{E}\left\{\frac{S_n^2}{n}\right\} &= \left|\int_{-1}^1 h_n(t)\ d\mu_f(t)\right| \leq \left|\int_{-1}^0 h_n(t)\ d\mu_f(t)\right| + \left|\int_0^1 h_n(t)\ d\mu_f(t)\right| \\ &\leq \int_{-1}^1 d\mu_f(t) + \int_0^1 \frac{1+t}{1-t} d\mu_f(t) < +\infty. \end{aligned}$$

Furthermore, by the dominated convergence theorem

$$\lim_{n\to\infty} \int_{-1}^0 h_n(t)\ d\mu_f(t) = \int_{-1}^0 \lim_{n\to\infty} h_n(t)\ d\mu_f(t) = \int_{-1}^0 \frac{1+t}{1-t} d\mu_f(t)$$

and by the monotone convergence theorem

$$\lim_{n\to\infty} \int_0^1 h_n(t)\ d\mu_f(t) = \int_0^1 \lim_{n\to\infty} h_n(t)\ d\mu_f(t) = \int_0^1 \frac{1+t}{1-t} d\mu_f(t).$$

Hence,

$$\lim_{n\to\infty} \mathbb{E}\left\{\frac{S_n^2}{n}\right\} = \lim_{n\to\infty} \int_{-1}^0 h_n(t)\ d\mu_f(t) + \lim_{n\to\infty} \int_0^1 h_n(t)\ d\mu_f(t) = \int_{-1}^1 \frac{1+t}{1-t} d\mu_f(t).$$

□

1.2.5.2 Central limit theorem for reversible Markov chain

Theorem 1.2.2. *(Kipnis - Varadhan, 1986) Assume that $(X_k)_{k\in\mathbb{Z}}$ is a stationary ergodic reversible Markov chain and $f \in L_0^2(\mu)$ satisfies*

$$\int_{-1}^1 \frac{1+t}{1-t} d\mu_f < +\infty \tag{1.28}$$

then the sequence

$$\frac{1}{\sqrt{n}}S_n \xrightarrow{\;D\;} \mathcal{N}\left(0,\sigma_f^2\right) \tag{1.29}$$

where $S_n = \sum_{k=0}^{n-1} f(X_k)$ *and* $\sigma_f^2 = \int_{-1}^{1} \frac{1+t}{1-t} d\mu_f(t)$.

Proof. For any $\varepsilon > 0$, then $\|P\|_{L^2(\mu)} < 1+\varepsilon$. It follows that $(1+\varepsilon)I - P$ invertible and denoted by

$$[(1+\varepsilon)I - P]^{-1} = \varphi_\varepsilon(P) = [\varphi_\varepsilon(P)]^* \tag{1.30}$$

and there exists $u_\varepsilon \in L^2(\mu)$ such that

$$(1+\varepsilon)u_\varepsilon - Pu_\varepsilon = f. \tag{1.31}$$

We will investigate the behavior of u_ε as $\varepsilon \to 0$. Put

$$f_\varepsilon = f - \varepsilon u_\varepsilon \tag{1.32}$$

then

$$Pu_\varepsilon - u_\varepsilon + f_\varepsilon = 0 \tag{1.33}$$

and put

$$M_n^\varepsilon = \sum_{k=0}^{n-1} [u_\varepsilon(X_{k+1}) - u_\varepsilon(X_k) + f_\varepsilon(X_k)] \tag{1.34}$$

then for each $\varepsilon > 0$, M_n^ε is a martingale with respect to $\mathcal{F}_n = \sigma(X_n, X_{n-1}, \ldots)$. Indeed, by using the fact

$$Ph(X_k) = \mathbb{E}\left\{h(X_{k+1})/X_k\right\} \tag{1.35}$$

for any function $h \in L^2(\mu)$, from (1.33) we have

$$\begin{aligned}
\mathbb{E}\left\{M_{n+1}^\varepsilon/\mathcal{F}_n\right\} &= M_n^\varepsilon + \mathbb{E}\left\{\left[u_\varepsilon(X_{n+1}) - u_\varepsilon(X_n) + f_\varepsilon(X_n)\right]/\mathcal{F}_n\right\} \\
&= M_n^\varepsilon + Pu_\varepsilon(X_n) - u_\varepsilon(X_n) + f_\varepsilon(X_n) \\
&= M_n^\varepsilon.
\end{aligned}$$

Now, for each $\varepsilon > 0$, then S_n is decomposed as follows

$$S_n = M_n^\varepsilon + \xi_n^\varepsilon + \eta_n^\varepsilon, \tag{1.36}$$

where

$$\begin{aligned}
\xi_n^\varepsilon &= -\sum_{k=0}^{n-1} \left[u_\varepsilon(X_{k+1}) - u_\varepsilon(X_k)\right], \\
\eta_n^\varepsilon &= \sum_{k=0}^{n-1} \left[f(X_k) - f_\varepsilon(X_n)\right] = \sum_{k=1}^{n} \varepsilon u_\varepsilon(X_k).
\end{aligned}$$

The next step we will show that S_n can be written as

$$S_n = M_n + \xi_n \tag{1.37}$$

where M_n is a martingale with respect to $\mathcal{F}_n$ and $\lim_{n\to\infty} \dfrac{1}{n}\mathbb{E}\{\xi_n^2\} = 0$.

Lemma 1.2.6. *For each $n \geq 1$,*

$$\lim_{\varepsilon\to 0} M_n^\varepsilon = M_n \text{ exists in } L^2(\mu). \tag{1.38}$$

Proof. Since M_n^ε is a martingale with stationary increments, to show that M_n^ε has a limit in $L^2(\mu)$, it is suffient to check that

$$\begin{aligned}\lim_{\varepsilon\to 0} M_1^\varepsilon &= \lim_{\varepsilon\to 0}\{u_\varepsilon(X_1) - u_\varepsilon(X_0) + f_\varepsilon(X_0)\} \\ &= \lim_{\varepsilon\to 0}\{u_\varepsilon(X_1) - Pu_\varepsilon(X_0)\} \text{ exists in } L^2(\mu).\end{aligned} \tag{1.39}$$

Since $L^2(\Omega,\mu)$ is complete, we need to check only that $\{u_\varepsilon(X_1) - Pu_\varepsilon(X_0)\}_\varepsilon$ is a Cauchy sequence as follows

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \mathbb{E}\left\{[(u_{\varepsilon_1} - u_{\varepsilon_2})(X_1) - P(u_{\varepsilon_1} - u_{\varepsilon_2})(X_0)]^2\right\} = 0. \tag{1.40}$$

We have

$$\begin{aligned}\mathbb{E}\left\{[u(X_1) - Pu(X_0)]^2\right\} &= \mathbb{E}\left\{u^2(X_1) - 2u(X_1)(Pu)(X_0) + (Pu)^2(X_0)\right\} \\ &= \mathbb{E}\{u^2(X_1) - 2\mathbb{E}\left\{u(X_1)Pu(X_0)/X_0\right\} + (Pu)^2(X_0)\} \\ &= \mathbb{E}\left\{u^2(X_1)\right\} - \mathbb{E}\left\{(Pu)^2(X_0)\right\} \\ &= \left\langle u, (I-P^2)u\right\rangle.\end{aligned} \tag{1.41}$$

Applying the above formula for $u = u_{\varepsilon_1} - u_{\varepsilon_2}$, then (1.40) becomes

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \left\langle u_{\varepsilon_1} - u_{\varepsilon_2}, (I-P^2)(u_{\varepsilon_1} - u_{\varepsilon_2})\right\rangle = 0. \tag{1.42}$$

From equation (1.31), we have

$$u_\varepsilon = [(1+\varepsilon)I - P]^{-1} f = \varphi_\varepsilon(P)f.$$

Using the reversibility of the chain i.e $P = P^*$, we have

$$\begin{aligned}\left\langle u_{\varepsilon_1} - u_{\varepsilon_2}, (I-P^2)(u_{\varepsilon_1} - u_{\varepsilon_2})\right\rangle &= \left\langle (I-P^2)\left[\varphi_{\varepsilon_1}(P) - \varphi_{\varepsilon_2}(P)\right]^2 f, f\right\rangle \\ &= \langle \Phi(P)f, f\rangle\end{aligned}$$

where $\Phi(P) = (I-P^2)\left[\varphi_{\varepsilon_1}(P) - \varphi_{\varepsilon_2}(P)\right]^2$. We recall

$$\langle P^m f, f\rangle = \int_{-1}^{1} t^m d\mu_f(t) \tag{1.43}$$

and more generally that

$$\langle \phi(P)f, f\rangle = \int_{-1}^{1} \phi(t) d\mu_f(t), \quad \forall \phi \in L^2(\mu_f). \tag{1.44}$$

Since $\Phi(t) = (1-t^2)\left(\frac{1}{1+\varepsilon_1 - t} - \frac{1}{1+\varepsilon_2 - t}\right)^2 \in L^2(\mu_f)$ then by (1.44)

$$\langle \Phi(P)f, f\rangle = \int_{-1}^{1} \Phi(t) d\mu_f(t)$$

Without losing the generality, we can assume that $\varepsilon_2 \geq \varepsilon_1 \geq 0$, one has

$$\begin{aligned}\Phi(t) &= \frac{(\varepsilon_2-\varepsilon_1)^2(1-t^2)}{(1+\varepsilon_1-t)^2(1+\varepsilon_2-t)^2} \leq \frac{\varepsilon_2^2(1-t^2)}{(1-t)^2\varepsilon_2^2} \\ &\leq \frac{1+t}{1-t},\end{aligned}$$

then (1.28) ensures that $\frac{1+t}{1-t}$ is integrable with respect to $d\mu_f$. By the dominated convergence theorem, we obtain

$$\lim_{\varepsilon_1,\varepsilon_2\to 0}\left\langle u_{\varepsilon_1}-u_{\varepsilon_2},(I-P^2)(u_{\varepsilon_1}-u_{\varepsilon_2})\right\rangle = \lim_{\varepsilon_1,\varepsilon_2\to 0}\int_{-1}^{1}\Phi(t)d\mu_f(t)=0$$

which completes (1.38). □

Lemma 1.2.7. *We have*

$$\lim_{\varepsilon\to 0}\varepsilon\|u_\varepsilon\|^2_{L^2(\mu)}=0 \tag{1.45}$$

and for each $n \geq 1$,

$$\lim_{\varepsilon\to 0}\eta_n^\varepsilon = 0 \textit{ in } L^2(\mu). \tag{1.46}$$

Proof. By the definition of η_n^ε it is easy to see that (1.45) implies (1.46). To obtain (1.45), we consider $\varphi_\varepsilon^2(t)=\frac{1}{(1+\varepsilon-t)^2}\in L^2(\mu_f)$. By (1.44) one has

$$\varepsilon\left\langle u_\varepsilon,u_\varepsilon\right\rangle = \varepsilon\left\langle\varphi_\varepsilon^2(P)f,f\right\rangle=\int_{-1}^{1}\frac{\varepsilon}{(1+\varepsilon-t)^2}d\mu_f(t)<\int_{-1}^{1}\frac{1}{1-t}d\mu_f(t)<+\infty.$$

By the dominated convergence theorem, we have thus proved (1.45). □

Now in (1.36), it remains ξ_n^ε. It will be treated by the following lemma

Lemma 1.2.8. *For each $n \geq 1$,*

$$\lim_{\varepsilon\to 0}\xi_n^\varepsilon=\xi_n \textit{ exists in } L^2(\mu) \tag{1.47}$$

and

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\{|\xi_n|^2\}=0. \tag{1.48}$$

Proof. Since $S_n = M_n^\varepsilon+\xi_n^\varepsilon+\eta_n^\varepsilon$ and is independent of $\varepsilon>0$, (1.38) and (1.46) imply (1.47). Furthermore $\xi_n = S_n - M_n$, hence for every $\varepsilon > 0$

$$\xi_n = M_n^\varepsilon - M_n + \xi_n^\varepsilon+\eta_n^\varepsilon.$$

Since $M_n^\varepsilon - M_n$ is a martingale with stationary increments, using Cauchy-Schwarz's inequality

$$\begin{aligned}\frac{1}{n}\mathbb{E}\left\{|\xi_n|^2\right\} &= \frac{1}{n}\mathbb{E}\left\{|M_n^\varepsilon-M_n+\xi_n^\varepsilon+\eta_n^\varepsilon|^2\right\} \\ &\leq \frac{3}{n}\mathbb{E}\left\{|M_n^\varepsilon-M_n|^2\right\}+\frac{3}{n}\mathbb{E}\left\{|\xi_n^\varepsilon|^2\right\}+\frac{3}{n}\mathbb{E}\left\{|\eta_n^\varepsilon|^2\right\}\end{aligned}$$

$$= \quad 3\mathbb{E}\left\{|M_1^\varepsilon - M_1|^2\right\} + \frac{3}{n}\mathbb{E}\left\{|\xi_n^\varepsilon|^2\right\} + \frac{3}{n}\mathbb{E}\left\{|\eta_n^\varepsilon|^2\right\}.$$

Using (1.38)

$$\lim_{\varepsilon\to 0}\mathbb{E}\left\{|M_1^\varepsilon - M_1|^2\right\} = 0.$$

Therefore it is sufficient to choose $\varepsilon = 1/n$ and then show that

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left\{|\xi_n^{1/n}|^2\right\} = 0 \text{ and } \lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left\{|\eta_n^{1/n}|^2\right\} = 0.$$

Clearly,

$$\begin{aligned}
\mathbb{E}\left\{|\xi_n^{1/n}|^2\right\} &= \mathbb{E}\left\{|u_{1/n}(X_n) - u_{1/n}(X_0)|^2\right\} \le \mathbb{E}\left\{\left(|u_{1/n}(X_n)| + |u_{1/n}(X_0)|\right)^2\right\} \\
&\le \mathbb{E}\left\{2|u_{1/n}(X_n)|^2 + 2|u_{1/n}(X_0)|^2\right\} \le 4\left\{\mathbb{E}|u_{1/n}(X_0)|^2\right\} \\
&= 4\left\langle u_{1/n}, u_{1/n}\right\rangle = o(n).
\end{aligned}$$

by (1.45). Similarly

$$\begin{aligned}
\mathbb{E}\left\{\left|\eta_n^{1/n}\right|^2\right\} &= \mathbb{E}\left\{\left[\frac{1}{n}\sum_{k=0}^{n-1} u_{1/n}(X_k)\right]^2\right\} \le \frac{1}{n}\mathbb{E}\left\{\left[\sum_{k=0}^{n-1}|u_{1/n}(X_k)|\right]^2\right\} \\
&\le \frac{1}{n}\mathbb{E}\left\{n\sum_{k=0}^{n-1}|u_{1/n}(X_k)|^2\right\} \le \mathbb{E}\left\{|u_{1/n}(X_0)|^2\right\} \\
&= \left\langle u_{1/n}, u_{1/n}\right\rangle = o(n).
\end{aligned}$$

□

We now return the proof of theorem 1.2.2. Combine (1.38), (1.46) and (1.47) then by (1.36) one has

$$\frac{1}{\sqrt{n}}S_n = \frac{1}{\sqrt{n}}M_n + \frac{1}{\sqrt{n}}\xi_n \tag{1.49}$$

with M_n is a martingale with respect to $\mathcal{F}_n$ and $\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left\{|\xi_n|^2\right\} = 0$. Therefore, it remains to show that

$$\frac{1}{\sqrt{n}}M_n \xrightarrow{D} \mathcal{N}\left(0, \sigma_f^2\right) \tag{1.50}$$

to complete the proof of theorem 1.2.2. Set $Y_n = M_n - M_{n-1}$ with $M_0 = 0$, then $(Y_n)_{n\ge 1}$ is a stationary ergodic sequence and by (1.41)

$$\begin{aligned}
Var(Y_1) &= \mathbb{E}\left\{|M_1|^2\right\} - \mathbb{E}\left\{M_1\right\}^2 = \lim_{\varepsilon\to 0}\mathbb{E}\left\{|M_1^\varepsilon|^2\right\} - \lim_{\varepsilon\to 0}\mathbb{E}\left\{M_1^\varepsilon\right\}^2 \\
&= \lim_{\varepsilon\to 0}\left\langle u_\varepsilon, (I - P^2)u_\varepsilon\right\rangle - \lim_{\varepsilon\to 0}\mathbb{E}\left\{u_\varepsilon(X_1) - u_\varepsilon(X_0) + f_\varepsilon(X_0)\right\}^2 \\
&= \int_{-1}^{1}\frac{1-t}{1+t}d\mu_f(t) - 0 = \sigma_f^2.
\end{aligned}$$

By the hypothesis (1.28), $Var(Y_1)$ is finite. Moreover,

$$\mathbb{E}\left\{Y_n/Y_{n-1}, \ldots, Y_1\right\} = \mathbb{E}\left\{M_n - M_{n-1}/X_{n-1}, \ldots, X_0\right\} = 0.$$

Hence, $(Y_n)_{n\geq 1}$ satisfies Billingsley's theorem which is stated and proved on page 51 (theorem 2.3.1), we recall it for convenience: *Suppose the sequence of* $(\tilde{X}_n)_{n\geq 1}$ *be stationary and ergodic such that* $Var\left\{\tilde{X}_1\right\} = \mathbb{E}\{\tilde{X}_1^2\}$ *is finite and*

$$\mathbb{E}\{\tilde{X}_n/\tilde{X}_1,\dots,\tilde{X}_{n-1}\} = 0, \quad a.s. \tag{1.51}$$

Then the distribution of $\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\tilde{X}_k$ *tends to the normal distribution with mean* 0 *and variance* $\mathbb{E}\{\tilde{X}_1^2\}$.

Applying this theorem for $(Y_n)_{n\geq 1}$, we have

$$\frac{1}{\sqrt{n}}M_n = \frac{1}{\sqrt{n}}\sum_{k=1}^{n} Y_k \xrightarrow{D} \mathcal{N}\left(0,\sigma_f^2\right)$$

with $\sigma_f^2 = \mathbb{E}\left\{Y_1^2\right\} = \int_{-1}^{1}\frac{1-t}{1+t}d\mu_f(t)$ which completes (1.50). □

1.3 Spectral measure with values in operator's space

In the sequel we consider the general case of a bounded operator P is self-adjoint, i.e $P = P^\star$ (not necessary a Markov operator).

1.3.1 Spectral measure with values in operator's space

We recall that

$$S(\mu_f) = \{t : \mu_f\,[t-\varepsilon; t+\varepsilon] > 0,\ \forall\varepsilon > 0\}.$$

In this section we will study the relationship between $S(\mu_f)$ and spectral measure with values in operator's space.

Proposition 1.3.1. *There exists* $f \in \mathcal{H}$ *such that for any* $g \in \mathcal{H}$ *the measure* μ_g *is absolutely continuous with respect to* μ_f.

We say that f has the maximal spectral type.

Proof. $\oplus$ Let $f' \in \mathcal{H}(P,f)^\perp$ then $\langle P^m f, P^n f'\rangle_{L^2(\mu)} = 0,\ \ \forall m,n\geq 0$.

There exists finite positve measures $\mu_f, \mu_{f'}, \mu_{f+f'}$ such that:

$$\begin{aligned}
\hat{\mu}_f(u) &= \int e^{iut}d\mu_f(t)\\
\hat{\mu}_{f'}(u) &= \int e^{iut}d\mu_{f'}(t)\\
\hat{\mu}_{f+f'}(u) &= \int e^{iut}d\mu_{f+f'}(t).
\end{aligned}$$

We have

$$\hat{\mu}_{f+f'}(u) = \sum_{m=0}^{\infty}\frac{(iu)^m}{m!}\left\langle P^m(f+f'), f+f'\right\rangle = \sum_{m=0}^{\infty}\frac{(iu)^m}{m!}(\langle P^m f, f\rangle + \langle P^m f', f'\rangle)$$

$$= \sum_{m=0}^{\infty} \frac{(iu)^m}{m!} \langle P^m f, f\rangle + \sum_{m=0}^{\infty} \frac{(iu)^m}{m!} \left\langle P^m f', f'\right\rangle = \hat{\mu}_f(u) + \hat{\mu}_{f'}(u)$$

then $\mu_{f+f'} = \mu_f + \mu_{f'}$.
$\oplus$ Select $g_1, g_2, \ldots, g_n, \ldots$ a complete orthonormal set in $\mathcal{H}$.

$\circledcirc$ Put $f_1 = g_1$, let P_{f_1} be the orthonormal projection on $\mathcal{H}(P, f_1)$.

$\circledcirc$ Put $f_2 = g_2 - P_{f_1}(g_1)$, let P_{f_2} be the orthonormal projection on $\mathcal{H}(P, f_2)$.

....

$\circledcirc$ Put $f_{r+1} = g_{r+1} - P_{f_1}(g_{r+1}) - P_{f_2}(g_{r+1}) - \ldots - P_{f_r}(g_{r+1})$, let $P_{f_{r+1}}$ be the orthonormal projection on $\mathcal{H}(P, f_{r+1})$.

....

We see that each $\mathcal{H}(P, f_i)$ is invariant under all $\sum\limits_{k=0}^{n} a_k P^k$ and $\mathcal{H}(P, f_i) \perp \mathcal{H}(P, f_j)$ if $i \neq j$.

Hence,

$$\mathcal{H} = \mathcal{H}(P, f_1) \bigoplus \mathcal{H}(P, f_2) \bigoplus \ldots \bigoplus \mathcal{H}(P, f_n) \bigoplus \ldots$$

since for each n, $g_n \in \mathcal{H}(P, f_1) \bigoplus \mathcal{H}(P, f_2) \bigoplus \ldots \bigoplus \mathcal{H}(P, f_n)$.

Set

$$f = \frac{1}{2} f_1 + \frac{1}{2^2} f_2 + \ldots + \frac{1}{2^n} f_n + \ldots = \sum_{k=1}^{\infty} \frac{1}{2^k} f_k \in \mathcal{H}.$$

We have

$$\mu_f = \frac{1}{2} \mu_{f_1} + \frac{1}{2^2} \mu_{f_2} + \ldots + \frac{1}{2^n} \mu_{f_n} + \ldots = \sum_{k=1}^{\infty} \frac{1}{2^k} \mu_{f_k} < \infty$$

and so $\mu_f(A) = 0$ if and only if $\mu_{f_i}(A) = 0$ for any $i = 1, 2, 3, \ldots$. Hence, μ_{f_i} is absolute continuous with respect to μ_f (denote $\mu_{f_i} \ll \mu_f$).

Moreover, for any $g \in \mathcal{H}$, we can decomposite g followed by $f_1, f_2, f_3, \ldots$. Therefore μ_g is absolutely continuous with respect to μ_f and so f has maximal spectral type. $\square$

Lemma 1.3.1. *If f and f' have maximal spectral type, then $S(\mu_f) = S(\mu_{f'})$ (Denoted $S(P)$ in the sequel).*

Proof. If f, f' have maximal spectral type then $\mu_f \ll \mu_{f'}$ and $\mu_{f'} \ll \mu_f$. So, μ_f *and* $\mu_{f'}$ are equivalent.

We recall that

$$\begin{aligned} S(\mu_f) &= \{t : \mu_f [t-\varepsilon; t+\varepsilon] > 0, \ \forall \varepsilon > 0\} \\ S(\mu_{f'}) &= \{t : \mu_{f'} [t-\varepsilon; t+\varepsilon] > 0, \ \forall \varepsilon > 0\} \end{aligned}$$

Suppose $t \notin S(\mu_f)$, then there exists $\varepsilon > 0$ such that $\mu_f [t-\varepsilon; t+\varepsilon] = 0$. It follows that $\mu_{f'} [t-\varepsilon; t+\varepsilon] = 0$ and so $t \notin S(\mu_{f'})$.

In the converse, $t \notin S(\mu_{f'})$ then $t \notin S(\mu_f)$. Hence, $S(\mu_f) = S(\mu_{f'}) = S(P)$. $\square$

Theorem 1.3.1. *The map* $\sum_{\ell=0}^{n} a_k t^k \mapsto \sum_{\ell=0}^{n} a_k P^k$ *can be extended to an isometry from the space* $\mathcal{C}(S(P))$ *of continuous functions on* $S(P)$ *with norm of uniform convergence, to the space* $\mathcal{L}(\mathcal{H})$ *of linear bounded operator with the usual operator's norm.*

Proof. $\oplus$ For any polynomial $R(X) = \sum_{\ell=0}^{n} a_k X^k$, we define the operator $R(P)$ on $\mathcal{H}$ by $R(P) = \sum_{\ell=0}^{n} a_k P^k$.

Let us consider the restriction of $R(P)$ on $\mathcal{H}(P, f)$. If

$$g = \Psi(Q) = \sum_{k=0}^{n} b_k P^k f$$

then

$$R(P)g = \sum_{\ell=0}^{n} a_\ell P^\ell \left(\sum_{k=0}^{n} b_k P^k f \right) = \sum_{\ell=0}^{n} \sum_{k=0}^{n} a_\ell b_k P^{\ell+k} f = \Psi(RQ).$$

For any $g = \Psi(h)$ with $h \in L^2(\mu_f)$, there exists $(Q_n)_{n\geq 1} \to h$ such that

$$R(P)\Psi(Q_n) = \Psi(RQ_n), \quad \forall n \geq 1.$$

For $n \to \infty$, by the continuity and linearity of Ψ, we obtain

$$R(P)g = \Psi(Rh).$$

Hence,

$$\begin{aligned} \|R(P)\|_{\mathcal{H}(P,f)} &= \sup_{\|g\|_{\mathcal{H}(P,f)}=1} \|R(P)g\|_{\mathcal{H}(P,f)} = \sup_{\|\Psi(h)\|_{\mathcal{H}(P,f)}=1} \|\Psi(Rh)\|_{\mathcal{H}(P,f)} \\ &= \sup_{\|h\|_{L^2(\mu_f)}=1} \|Rh\|_{L^2(\mu_f)} \leq \sup_{t\in S(\mu_f)} |R(t)|. \end{aligned}$$

We will prove that this inequality is equality. There exists $t_0 \in \overline{S(\mu_f)}$ such that

$$|R(t_0)| = \sup_{t\in S(\mu_f)} |R(t)|$$

and for each $1 \leq n \in \mathbb{N}$, let $t_n \in S(\mu_f)$ such that $|t_n - t_0| < \dfrac{1}{n}$.

Choose $h_n = \dfrac{1}{\sqrt{c_n}} \mathbf{1}_{B(t_n,1/n)}$ with $c_n = \displaystyle\int_{t_n-\frac{1}{n}}^{t_n+\frac{1}{n}} d\mu_f > 0$ since $t_n \in S(\mu_f)$ and $B(t_n, 1/n)$ be the open balls have radius $1/n$ and center at t_n, then $\|h_n\|_{L^2(\mu_f)} = 1$. By computing,

$$\|Rh_n\|_{L^2(\mu_f)} = \frac{1}{\sqrt{c_n}} \sqrt{\int_{t_n-1/n}^{t_n+1/n} R^2(t) d\mu_f}, \quad \forall n \geq 1.$$

For n is large enough,

$$\|Rh_n\|_{L^2(\mu_f)} \approx |R(t_n)| \approx |R(t_0)|.$$

Hence,

$$\sup_{\|h_n\|_{L^2(\mu_f)}} \|Rh_n\|_{L^2(\mu_f)} = |R(t_0)| = \sup_{t\in S(\mu_f)} |R(t)|$$

so,

$$\|R(P)\|_{\mathcal{H}(P,f)} = \sup_{\|h\|_{L^2(\mu_f)}} \|R(t)h\|_{L^2(\mu_f)} = \sup_{t\in S(\mu_f)} |R(t)|.$$

Suppose that $f \in \mathcal{H}$ be a maximal spectral type , then

$$\|R(P)\|_{\mathcal{H}(P,f)} = \sup_{t\in S(\mu_f)} |R(t)| = \sup_{g\in\mathcal{H}} \sup_{t\in S(\mu_g)} |R(t)| = \sup_{g\in\mathcal{H}} \|R(P)\|_{\mathcal{H}(P,g)} = \|R(P)\|_{\mathcal{H}}$$

and hence $\|R(P)\|_{\mathcal{H}} = \sup_{t\in S(P)} |R(t)|$.

$\oplus$ Let $\mathcal{P}$ be the linear subspace of $\mathcal{C}(\mathcal{S}(P))$ consiting all polynomials, where $\mathcal{C}(\mathcal{S}(P))$ is the space of continuous functions on $\mathcal{S}(P) \subset \mathbb{R}$.

Define:

$$\begin{array}{rccc} \phi: & \mathcal{P} & \longrightarrow & \mathcal{L}(\mathcal{H}) \\ & R & \longmapsto & R(P). \end{array}$$

then ϕ is a linear transformation such that $\phi(QR) = \phi(Q)\phi(R)$ for all $Q, R \in \mathcal{P}$ and $\|\phi(R)\|_{\mathcal{L}(\mathcal{H})} = \|R(P)\|_{\mathcal{L}(\mathcal{H})} = \sup_{t\in S(P)} |R(t)| = \|R\|_{\mathcal{C}(\mathcal{S}(P))}$. So, ϕ is isometry.

Moreover, since $\mathcal{P}$ is dense in $\mathcal{C}(\mathcal{S}(P))$, ϕ can be extended to an isometry from $\mathcal{C}(\mathcal{S}(P))$ with uniform convergence, to the space $\mathcal{L}(\mathcal{H})$ of linear bounded operator on $\mathcal{H}$, with the norm of operators since $\mathcal{L}(\mathcal{H})$ is a complete space. □

Usually, Θ is denoted as a Radon measure dE with values in $\mathcal{L}(\mathcal{H})$:

$$\Theta(h) = \int_{\mathcal{S}(P)} h(t)dE(t). \tag{1.52}$$

Proposition 1.3.2. *For any $f \in \mathcal{H}$ and for any $h \in \mathcal{C}(\mathcal{S}(P))$, we have*

$$\int \Theta(h)f \cdot \bar{f}\, d\mu = \int h\, d\mu_f. \tag{1.53}$$

This equality is usually denoted by

$$\langle dE(t)f, f\rangle = d\mu_f(t) \tag{1.54}$$

Proof. Denote $\mathcal{P}$ be the linear subspace of $\mathcal{C}(\mathcal{S}(P))$ consisting all polynomials.

If $h = R(t) \in \mathcal{P}$ then $\Theta(h) = \sum\limits_{k=0}^{n} a_k P^k$. Hence,

$$\int \Theta(h)f \cdot \bar{f} d\mu = \sum_{k=0}^{n} a_k \gamma(k) = \sum_{k=0}^{n} a_k \int t^k d\mu_f = \int h d\mu_f.$$

If $h \in \mathcal{C}(\mathcal{S}(P))$ then there exists a sequence $(R_n(t))_n$ in $\mathcal{P}$ which converges to h and

$$\int \Theta(R_n(t))f \cdot \bar{f} d\mu = \int R_n(t) d\mu_f.$$

For $n \to \infty$, by the dominated convergence theorem, it follows that

$$\int \Theta(h)f \cdot \bar{f} d\mu = \int h d\mu_f.$$

. □

1.3.2 Approximate eigenvalues

Definition 1.3.1. *A bounded operator A is normal if $A \cdot A^* = A^* \cdot A$*

Definition 1.3.2. *The spectrum of P is the set $\Sigma(P)$ of $\lambda \in \mathbb{C}$ such that $P - \lambda I$ is not invertible (as a bounded Hilbert operator of $L^2(\mu)$). The resolvent set is its complementary $\Omega(P) = \mathbb{C} \setminus \Sigma(P)$.*

Denote $\rho(P) = \sup_{\lambda \in \Sigma(P)} |\lambda|$ the spectral radius.

The aim of this paragraph is to prove the following theorem

Theorem 1.3.2. *We have $\Sigma(P) = S(P)$.*

To prove this theorem, we prove that these two sets are equal to the set of approximate eigenvalues, defined as follows

Definition 1.3.3. *A complex number $\lambda \in \mathbb{C}$ is an approximate eigenvalue if there exists $(f_n)_n$ such that $\|f_n\|_{L^2(\mu)} = 1$ and $\|(P - \lambda I)f_n\|_{L^2(\mu)} \to 0$ for $n \to \infty$.*

Denote $\mathcal{V}(P)$ the set of approximate eigenvalues.

Proposition 1.3.3. *We have $\Sigma(P) = \mathcal{V}(P)$.*

Proof. We need two steps:

The first step is to prove that $\mathcal{V}(P) \subset \sum(P)$. Let $\lambda \in \mathcal{V}(P)$. If $\lambda \in \Omega(P)$ then $P - \lambda I$ is inversible. For any $f \in L^2(\mu)$,

$$\|f\| = \|(P - \lambda I)^{-1}(P - \lambda I)f\| \leq \|(P - \lambda I)^{-1}\|\|(P - \lambda I)f\|$$

and so

$$\|(P - \lambda I)f\| \geq \|(P - \lambda I)^{-1}\|^{-1}\|f\|.$$

This implies that $\|(P-\lambda I)f_n\| \geq \|(P-\lambda I)^{-1}\|^{-1} > 0$ for any $(f_n)_n$ such that $\|f_n\|_{L^2(\mu)} = 1$ and hence, $\lambda \notin \mathcal{V}(P)$. This is a contradiction! We deduce that $\lambda \in \sum(P)$ and therefore $V(P) \subset \sum(P)$.

And, the second step is to prove $\sum(P) \subset \mathcal{V}(\mathcal{P})$. Let $\lambda \in \mathbb{C}$ and $\lambda \notin \mathcal{V}(\mathcal{P})$, we will prove that $\lambda \notin \Sigma(P)$ by showing that $P - \lambda I$ is invertible. In the proof, we will use the fact that is for any normal operator A, then $\mathrm{Ker}(A)^\perp = \overline{\mathrm{Im}(A)}$.

We first prove that $\overline{\mathrm{Im}(P-\lambda I)} = L^2(\mu)$. Since P is normal and $(P-\lambda I)^* = P^* - \bar{\lambda} I$, then

$$\begin{aligned}(P-\lambda I)(P-\lambda I)^* &= (P-\lambda I)(P^*-\bar{\lambda} I) = PP^* - \lambda P^* - \bar{\lambda} P + \lambda\bar{\lambda} I \\ &= (P^*-\bar{\lambda} I)(P-\lambda I) = (P-\lambda I)^*(P-\lambda I)\end{aligned}$$

and hence $P-\lambda I$ is normal. It follows that $\mathrm{Ker}(P-\lambda I)^\perp = \overline{\mathrm{Im}(P-\lambda I)}$. Moreover,there exists $\varepsilon > 0$ such that $\forall f \in L^2(\mu)$ then

$$\|(P-\lambda I)f\| \geq \varepsilon \|f\|.$$

If $f \in \mathrm{Ker}(P-\lambda I)$ then $(P-\lambda I)f = 0$. We have

$$0 = \|(P-\lambda I)f\| \geq \varepsilon\|f\| \Longrightarrow \|f\| = 0 \Longrightarrow f = 0$$

Thus,

$$\mathrm{Ker}(P-\lambda I) = \{0\} \Longrightarrow \mathrm{Ker}(P-\lambda I)^\perp = \overline{\mathrm{Im}(P-\lambda I)} = L^2(\mu).$$

Now, we will prove $\mathrm{Im}(P-\lambda I) = L^2(\mu)$. Let $(g_n)_n$ be a sequence in $\mathrm{Im}(P-\lambda I)$ tending to $g \in \overline{\mathrm{Im}(P-\lambda I)}$. Then, there exists $(f_n)_n \subset \overline{\mathrm{Im}(P-\lambda I)}$ such that $g_n = (p-\lambda I)f_n$.

We have

$$\|g_m - g_n\| = \|(P-\lambda I)(f_m - f_n)\| \geq \varepsilon\|f_m - f_n\|, \quad \forall m, n \in \mathbb{N}.$$

Since $g_n \to g \in \overline{\mathrm{Im}(P-\lambda I)}$, then $(g_n)_n$ be a Cauchy sequence. And hence, $(f_n)_n$ be also a Cauchy sequence in $\overline{\mathrm{Im}(P-\lambda I)}$. Then, $\exists f \in \overline{\mathrm{Im}(P-\lambda I)} : f_n \to f$. By the continuity of $P-\lambda I$, we have

$$(P-\lambda I)f = (P-\lambda I)\lim_{n\to\infty} f_n = \lim_{n\to\infty}(P-\lambda I)f_n = \lim_{n\to\infty} g_n = g.$$

It follows that $g \in \mathrm{Im}(P-\lambda I)$, implies $\mathrm{Im}(P-\lambda I)$ is closed, and hence $\mathrm{Im}(P-\lambda I) = L^2(\mu)$.

Finally, we prove that $P-\lambda I$ is invertible. Since $\mathrm{Im}(P-\lambda I) = L^2(\mu)$, then $P-\lambda I$ is one to one onto $L^2(\mu)$. Thus, $P-\lambda I$ be a bijection and so there exists unique linear transformation $(P-\lambda I)^{-1}$ from $L^2(\mu)$ onto $L^2(\mu)$. We will show that $(P-\lambda I)^{-1}$ is also bounded. For any $g \in L^2(\mu)$, there exists $f \in L^2(\mu)$ such that

$$(P-\lambda I)f = g \Longrightarrow f = (P-\lambda I)^{-1}g.$$

Since

$$\|g\| = \|(P-\lambda I)f\| \geq \varepsilon\|f\| = \varepsilon\|(P-\lambda I)^{-1}g\|$$

then

$$\|(P-\lambda I)^{-1}g\| \leq \frac{1}{\varepsilon}\|g\|$$

and so

$$\|(P-\lambda I)^{-1}\| \leq \frac{1}{\varepsilon}.$$

Therefor, $P-\lambda I$ is invertible. It follows that $\lambda \notin \Sigma(P)$. We finish the proof.

□

Proposition 1.3.4. *We have* $S(P) = \mathcal{V}(P)$.

Proof. We need also two steps to prove this proposition.

Firstly, we prove that $S(P) \subset \mathcal{V}(P)$. Let $f \in \mathcal{H}$ and $t_0 \in S(\mu_f)$ and claim that $t_0 \in \mathcal{V}(P)$. It is sufficient to show that there exists a sequence $(f_n)_{n\geq 1} \subset L^2(\mu)$ such that $\|f_n\|_{L^2(\mu)} = 1$ and $\lim\limits_{n\to\infty} \|(P - t_0 I)f_n\|_{L^2(\mu)} = 0$.

Let $f_n = \frac{1}{\sqrt{c_n}}\mathbf{1}_{(t_0-1/n;t_0+1/n)}$ with $c_n = \int\limits_{t_0-1/n}^{t_0+1/n} d\mu_f > 0$. Then by computing, $\|f_n\|_{L^2(\mu_f)} = 1, \ \forall n \geq 1$. One has

$$
\begin{aligned}
\lim_{n\to\infty} \|(P - t_0 I)f_n\|_{L^2(\mu)} &= \lim_{n\to\infty} \left\langle \left(P^2 - 2t_0 P + t_0^2 I\right) f_n, f_n \right\rangle \\
&= \lim_{n\to\infty} \left(\langle P^2 f_n, f_n\rangle - 2t_0\langle P f_n, f_n\rangle + t_0^2 \langle f_n, f_n\rangle\right) \\
&= \lim_{n\to\infty} \frac{1}{c_n}\sqrt{\int\limits_{t_0-1/n}^{t_0+1/n} (t - t_0)^2 d\mu_f} = 0.
\end{aligned}
$$

And the second, we will prove that $\mathcal{V}(P) \subset S(P)$. Assume that f of maximal spectral type and $t_0 \notin S(\mu_f)$. For any $(f_n)_{n\geq 1}$ such that $\|f_n\|_{L^2(\mu)} = \int d\mu_{f_n} = 1$, we claim that $\|(P - t_0)f_n\|_{L^2(\mu)} \not\to 0$.

Since f is a maximal spectral type, then $\mu_{f_n} \ll \mu_f$ and there exists $(h_n)_{n\geq 1}$ such that $d\mu_{f_n} = h_n d\mu_f$ for any $n \geq 1$ and $\displaystyle\int h_n d\mu_f = 1$. Since $t_0 \notin S(\mu_f)$, there exists $\varepsilon > 0$ such that $\mu_f\,[t_0 - \varepsilon; t_0 + \varepsilon] = 0$. We have

$$
\begin{aligned}
\|(P - t_0 I)f_n\|^2_{L^2(\mu)} &= \int (t - t_0)^2 d\mu_{f_n} = \int (t - t_0)^2 h_n d\mu_f = \int\limits_{|t-t_0|>\varepsilon} (t - t_0)^2 h_n d\mu_f \\
&\geq \varepsilon^2 \int\limits_{|t-t_0|>\varepsilon} h_n d\mu_f = \varepsilon^2 > 0
\end{aligned}
$$

since $\displaystyle\int h_n d\mu = \int\limits_{|t-t_0|>\varepsilon} h_n d\mu_f = 1$. This shows that $t_0 \notin \Sigma(P)$ or $\mathcal{V}(P) \subset S(P)$. □

***Remark* 1.3.1.** *Let* λ *in the resolvent set. We have*

$$
(P - \lambda I)^{-1} = \int\limits_{\mathcal{S}(P)} \frac{1}{t - \lambda} dE(t). \tag{1.55}
$$

Indeed, let us consider $h(t) = \dfrac{1}{t - \lambda}$ with $\lambda \in \mathbb{C}/\mathcal{S}(P)$, it is a measurable bounded function. And $th(t)$ is also bounded.

One has $(t - \lambda)h(t) = 1$ and hence $(P - \lambda I)h(P) = I$. It follows that $(P - \lambda I)^{-1} = h(P) = \Theta(h)$ if $\lambda \in \Omega(P)$. Thus,

$$\big\langle (P-\lambda I)^{-1}f, f\big\rangle = \langle h(P)f, f\rangle = \int h(t)d\mu_f(t) = \int \frac{1}{t-\lambda}d\mu_f(t)$$

and we deduce that

$$(P-\lambda I)^{-1} = \int\limits_{\mathcal{S}(P)} \frac{1}{t-\lambda}dE(t).$$

***Remark* 1.3.2.** *We can use this theory to define* $\sqrt{I-P}$ *as follows*

$$\sqrt{I-P} = \int\limits_{\mathcal{S}(P)} \sqrt{1-t}dE(t). \tag{1.56}$$

***Remark* 1.3.3.** *Note that all preceding questions in subsection 1.3.1 and subsection 1.3.2 are still valid if* T *is unitary operator:* $T^* = T^{-1}$*, exepted remark 1.3.2 because* $z \longmapsto \sqrt{z}$ *is not defined on* $\mathbb{C}$*. We can prove that* $\Sigma(T)$ *is a closed subset of* $S^1 = \{z \in \mathbb{C};\ |z| = 1\}$*.*

Chapter 2

The proofs of Central limit theorem for martingales without Fourier analysis

2.1 Introduction

The main aim of this chapter is to use a new way without Fourier analysis to obtain again the CLT for martingales. About the CLT for martingales, they are very classical, we can find out in many works of Billingsley (1961, [4]), Brown (1971, [8]),... So, in our works here, we are just interested in the method to obtain again theorem. What is the method?

Let's us begin with an elementary proof of the CLT of Trotter in his paper [48] in 1959. In there, Trotter used operator's method to obtain the CLT for indentically independent distributed (iid) variables and non iid random variables. That is, for any function $f \in \mathcal{C}_B$, the set of bounded continuous functions, he introduced a linear operator associated to random variable (rv's) X with distribution function F

$$T_X f(y) = \mathbb{E}\{f(x+y)\} = \int f(x+y) dF(x). \tag{2.1}$$

Then he used the fact

$$\lim_{n\to\infty} \|T_{X_n} f - T_X f\| = 0, \quad \forall f \in \mathcal{C}^2 \tag{2.2}$$

to prove that the sequence of random variables $X_1, X_2, \ldots, X_n, \ldots$ converge in distribution to random variable X.

In the sequel, we will use the similar way without operator to obtain again the CLT for the cases of iid, non iid variables. The point of our method is using Taylor's expansion of a function up to the second derivative. It is necessary to give the proof for independent cases in detail, because it is useful for martingale cases. For martingales, we will adapt also some ideas from Billingsley [4] and Brown [8]. We thank also to Lindeberg for his proof in [36], in there he used a similar way but he needed more conditions for random variables.

We review the fact that a sequence of random variables $(X_n)_{n\geq 1}$ converge in distribution to rv's X if and only if

$$\lim_{n\to\infty} \mathbb{E}\left\{f(X_n)\right\} = \mathbb{E}\left\{f(X)\right\} \tag{2.3}$$

for any function $f \in \mathcal{C}_B$, the set of the bounded continuous functions. However, we need more properties about function f. We think of the following lemma.

Lemma 2.1.1. *Suppose that* $\sup_n \left\{\mathbb{E}(X_n^2)\right\}$ *and* $\mathbb{E}\left\{X^2\right\}$ *are finite.*

For any function $f \in \mathcal{C}_K^2$*, the set of functions* $f \in \mathcal{C}^2$ *with support compact. Setting*

$$I_n(f) = \mathbb{E}\left\{f(X_n)\right\} - \mathbb{E}\left\{f(X)\right\}. \tag{2.4}$$

If $\lim_{n\to\infty} I_n(f) = 0$*, then* $(X_n)_{n\geq 1}$ *converge in distribution to rv's* X*.*

Proof. We decompose the proof into two steps:

Step 1. For any function $g \in \mathcal{C}_K$, there exists a sequence of functions $g_k \in \mathcal{C}_K^2$ such that $g_k \to g$ in L^∞. We have

$$\begin{aligned} I_n(g) &= \mathbb{E}\{g(X_n) - g(X)\} \\ &= \mathbb{E}\{g(X_n) - g_k(X_n)\} + \mathbb{E}\{g_k(X_n) - g_k(X)\} + \mathbb{E}\{g_k(X) - g(X)\} \\ &\leq 2\|g - g_k\|_\infty + I_n(g_k) \end{aligned}$$

so we get

$$|I_n(g)| \leq 2\|g - f_k\|_\infty + |I_n(f_k)|.$$

For $n \to \infty$ and then for $k \to \infty$, we will obtain $\lim_{n\to\infty} I_n(g) = 0$.

Step 2. For any function $h \in \mathcal{C}_B$. We claim that $\lim_{n\to\infty} I_n(h) = 0$. Let

$$\sigma^2 = \max\left\{Var(X), \sup_n \left[Var(X_n)\right]\right\}.$$

By Chebyshev's inequality, for any $\varepsilon > 0$, there exists $M_\varepsilon > 0$ such that

$$\begin{aligned} \mathbb{P}\left\{|X_n| \geq M_\varepsilon\right\} &\leq \frac{Var(X_n)}{M_\varepsilon^2} \leq \frac{\sigma^2}{M_\varepsilon^2} = \varepsilon, \\ \mathbb{P}\left\{|X| \geq M_\varepsilon\right\} &\leq \frac{Var(X)}{M_\varepsilon^2} \leq \frac{\sigma^2}{M_\varepsilon^2} = \varepsilon. \end{aligned}$$

We define a continuous function g_ε on $\mathbb{R}$ by

$$g_\varepsilon(x) = \begin{cases} h(x) & if \quad x \in [-M_\varepsilon, M_\varepsilon] \\ g_0(x) & if \quad x \in [-M_\varepsilon - 1, -M_\varepsilon] \cup [M_\varepsilon, M_\varepsilon + 1] \\ 0 & if \quad otherwise, \end{cases}$$

where $|g_0(x)| \leq |h(x)|$. It is easy to see that $g_\varepsilon \in \mathcal{C}_K$.

We have

$$\begin{aligned} I_n(h) &= \mathbb{E}\{h(X_n) - h(X)\} \\ &= \mathbb{E}\{h(X_n) - g_\varepsilon(X_n)\} + \mathbb{E}\{g_\varepsilon(X_n) - g_\varepsilon(X)\} + \mathbb{E}\{g_\varepsilon(X) - h(X)\}. \end{aligned}$$

Since

$$\begin{aligned}
\mathbb{E}\{h(X_n) - g_\varepsilon(X_n)\} &= \mathbb{E}\{(h - g_\varepsilon)(X_n)\} \\
&= \mathbb{E}\left\{(h - g_\varepsilon)(X_n)\mathbf{1}_{\{|X_n| > M_\varepsilon\}}\right\} + \mathbb{E}\left\{(h - g_\varepsilon)(X_n)\mathbf{1}_{\{|X_n| \le M_\varepsilon\}}\right\} \\
&\le \|h - g_\varepsilon\|_\infty \mathbb{P}\{|X_n| > M_\varepsilon\} + 0 \cdot \mathbb{P}\{|X_n| \le M_\varepsilon\} \\
&\le \|h\|_\infty \cdot \varepsilon.
\end{aligned}$$

Similarly,

$$\mathbb{E}\{g_\varepsilon(X) - h(X)\} \le \|h\|_\infty \cdot \varepsilon$$

Therefore

$$|I_n(h)| \le 2\varepsilon\|h\|_\infty + |I_n(g_\varepsilon)| \text{ implies } \lim_{n\to\infty} |I_n(h)| \le 2\varepsilon\|h\|_\infty$$

For $\varepsilon \to 0$, we get the desired result that is $\lim_{n\to\infty} I_n(h) = 0$. That means $(X_n)_{n\ge1}$ converges in distribution to rv's X.

□

2.2 CLT for sequence of independent variables

In this section, we will use lemma 2.1.1 to obtain the CLT for iid variables and non iid variables. This is the case of independent random variables, adapted some ideas from Trotter [48].

2.2.1 Indentically independent distributed variables

Theorem 2.2.1. *Consider a sequence $(X_n)_{n\ge1}$ of iid random variables. Assume that they are centered, and have finite variance σ^2. Then the distribution of $\frac{1}{\sqrt{n}}\sum_{k=1}^{n} X_k$ tends to the standard normal law $\mathcal{N}(0,1)$ when $n \to \infty$.*

Proof. Denote by $(Y_n)_{n\ge1}$ a sequence of iid gaussian random variables $\mathcal{N}(0;\sigma^2)$, independent of the first sequence. Put

$$\begin{aligned}
V_n &= \frac{X_1 + X_2 + \ldots + X_n}{\sqrt{n}}, \\
W_n &= \frac{Y_1 + Y_2 + \ldots + Y_n}{\sqrt{n}}.
\end{aligned}$$

For any function $f \in \mathcal{C}_B$. Setting

$$I_n(f) = \mathbb{E}\{f(V_n)\} - \mathbb{E}\{f(W_n)\}.$$

Since the distribution of $W_n \sim \mathcal{N}(0,1)$, the theorem 2.2.1 would be proved if we prove that $\lim_{n\to\infty} I_n(f) = 0$. However, by Lemma 2.1.1 above we need only to prove $\lim_{n\to\infty} I_n(f) = 0$ for any function $f \in \mathcal{C}_K^2$, the set of functions $f \in \mathcal{C}^2$ with support compact. Setting

$$U_k = (X_1 + X_2 + \ldots + X_k) + (Y_{k+1} + Y_{k+2} + \ldots + Y_n).$$

We have

$$\begin{aligned} f(V_n) - f(W_n) &= \sum_{k=1}^{n} \left[f\left(\frac{U_k}{\sqrt{n}}\right) - f\left(\frac{U_{k-1}}{\sqrt{n}}\right)\right] \\ &= \sum_{k=1}^{n} \left[f\left(Z_k + \frac{X_k}{\sqrt{n}}\right) - f\left(Z_k + \frac{Y_k}{\sqrt{n}}\right)\right], \end{aligned}$$

where $Z_k = \frac{X_1+X_2+\ldots+X_{k-1}}{\sqrt{n}} + \frac{Y_{k+1}+Y_{k+2}+\ldots+Y_n}{\sqrt{n}}$.

Also, by Taylor's expansions

$$\begin{aligned} f\left(Z_k + \frac{X_k}{\sqrt{n}}\right) &= f(Z_k) + f'(Z_k)\frac{X_k}{\sqrt{n}} + \frac{1}{2} f''(M_k)\frac{X_k^2}{n}, \\ f\left(Z_k + \frac{Y_k}{\sqrt{n}}\right) &= f(Z_k) + f'(Z_k)\frac{Y_k}{\sqrt{n}} + \frac{1}{2} f''(N_k)\frac{Y_k^2}{n}, \end{aligned}$$

for some M_k and N_k such that $|M_k - Z_k| \le |X_k|/\sqrt{n}$ and $|N_k - Z_k| \le |Y_k|/\sqrt{n}$. Thus, we have

$$\begin{aligned} f\left(Z_k + \frac{X_k}{\sqrt{n}}\right) - f\left(Z_k + \frac{Y_k}{\sqrt{n}}\right) &= f'(Z_k)\left(\frac{X_n}{\sqrt{n}} - \frac{Y_n}{\sqrt{n}}\right) + \frac{f''(M_k)X_k^2}{2n} - \frac{f''(N_k)Y_k^2}{2n} \\ &= f'(Z_k)\left(\frac{X_n}{\sqrt{n}} - \frac{Y_n}{\sqrt{n}}\right) + \frac{1}{2} f''(Z_k)\left(\frac{X_k^2}{n} - \frac{Y_k^2}{n}\right) \\ &\quad + \frac{f''(M_k) - f''(Z_k)}{2} \cdot \frac{X_k^2}{n} - \frac{f''(N_k) - f''(Z_k)}{2} \cdot \frac{Y_k^2}{n} \\ &= I_1 + I_2 + I_3 - I_4. \end{aligned}$$

By independence of the random variables X_k, Y_k, Z_k, and $\mathbb{E}\{X\} = \mathbb{E}\{Y\} = 0$, $\mathbb{E}\{X_k^2\} = \mathbb{E}\{Y_k^2\} = \sigma^2$, the expectation of $I_1 + I_2$ is null and the remainder is $I_3 - I_4$. Since $f \in \mathcal{C}_K^2$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|z - s| < \delta$ then $|f''(z) - f''(s)| < \varepsilon$. It follows that

$$|I_3 - I_4| \le \frac{1}{n}\|f''\|_\infty \left(X_k^2 \mathbb{1}_{\{|X_k| > \delta\sqrt{n}\}} + Y_k^2 \mathbb{1}_{\{|Y_k| > \delta\sqrt{n}\}}\right) + \frac{\varepsilon}{2n}\left(X_k^2 + Y_k^2\right).$$

Thus, the upper bound of $\mathbb{E}\left\{f\left(Z_k + \frac{X_k}{\sqrt{n}}\right) - f\left(Z_k + \frac{Y_k}{\sqrt{n}}\right)\right\}$ is

$$\frac{1}{n}\|f''\|_\infty \left(\mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k| > \delta\sqrt{n}\}}\right\} + \mathbb{E}\left\{Y_k^2 \mathbb{1}_{\{|Y_k| > \delta\sqrt{n}\}}\right\}\right) + \frac{\varepsilon}{2n}\left(\mathbb{E}\left\{X_k^2\right\} + \mathbb{E}\left\{Y_k^2\right\}\right).$$

Taking the sum on $k = 1, 2, \ldots, n$, we have

$$|I_n| \le \|f''\|_\infty \frac{1}{n}\sum_{k=1}^{n}\left(\mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k| > \delta\sqrt{n}\}}\right\} + \mathbb{E}\left\{Y_1^2 \mathbb{1}_{\{|Y_1| > \delta\sqrt{n}\}}\right\}\right) + \varepsilon\sigma^2.$$

For $n \to \infty$, we see that $\lim_{n\to\infty} |I_n(f)| \le \varepsilon\sigma^2$. Since ε as small as we need, the theorem 2.2.1 is proved.

$\square$

2.2.2 Non indentically distributed variables

Suppose $(X_n)_{n\geq 1}$ be a sequence of independent random variables which does not have the same distribution. Assume that $\mathbb{E}\{X_k\}=0$, $\mathbb{E}\left\{X_k^2\right\}=\sigma_k^2$ and denote $s_n^2=\sum\limits_{k=1}^n \sigma_k^2$, we have the theorem as follows

Theorem 2.2.2. *If for any $\delta>0$, we have*

$$\lim_{n\to\infty}\frac{1}{s_n^2}\sum_{k=1}^n \mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k|>\delta s_n\}}\right\}=0, \tag{2.5}$$

then the distribution of $\frac{1}{s_n}\sum\limits_{k=1}^n X_k$ tends to the standard normal law $\mathcal{N}(0,1)$ when $n\to\infty$. Condition (2.5) is called Lindeberg's condition.

Proof. Denote by $(\xi_n)_{n\geq 1}$ a sequence of iid gaussian random variables $\mathcal{N}(0;1)$, independent of the first sequence. Let $(Y_n)_{n\geq 1}$ be a sequence of random variables such that $Y_n=\sigma_n\xi_n$, independent of $(X_n)_{n\geq 1}$. Put

$$\begin{aligned} V_n &= \frac{X_1+X_2+\ldots+X_n}{s_n}, \\ W_n &= \frac{Y_1+Y_2+\ldots+Y_n}{s_n}. \end{aligned}$$

For any function $f\in\mathcal{C}_K^2$, we consider

$$I_n(f)=\mathbb{E}\{f(V_n)\}-\mathbb{E}\{f(W_n)\}.$$

Since the distribution of $W_n\sim\mathcal{N}(0,1)$, the theorem 2.2.2 would be proved if we prove that $\lim\limits_{n\to\infty} I_n(f)=0$.

Set $U_k=(X_1+X_2+\ldots+X_k)+(Y_{k+1}+Y_{k+2}+\ldots+Y_n)$. We have:

$$\begin{aligned} f(V_n)-f(W_n) &= \sum_{k=1}^n\left[f\left(\frac{U_k}{s_n}\right)-f\left(\frac{U_{k-1}}{s_n}\right)\right] \\ &= \sum_{k=1}^n\left[f\left(Z_k+\frac{X_k}{s_n}\right)-f\left(Z_k+\frac{Y_k}{s_n}\right)\right], \end{aligned}$$

where $Z_k=\frac{X_1+X_2+\ldots+X_{k-1}}{s_n}+\frac{Y_{k+1}+Y_{k+2}+\ldots+Y_n}{s_n}$.

Also, by Taylor's expansions

$$\begin{aligned} f\left(Z_k+\frac{X_k}{s_n}\right) &= f(Z_k)+f'(Z_k)\frac{X_k}{s_n}+\frac{1}{2}f''(M_k)\frac{X_k^2}{s_n^2}, \\ f\left(Z_k+\frac{Y_k}{s_n}\right) &= f(Z_k)+f'(Z_k)\frac{Y_k}{s_n}+\frac{1}{2}f''(N_k)\frac{Y_k^2}{s_n^2} \end{aligned}$$

for some M_k and N_k such that $|M_k - Z_k| \leq |X_k|/s_n$ and $|N_k - Z_k| \leq |Y_k|/s_n$. Thus, we have

$$\begin{aligned}
f\left(Z_k + \frac{X_k}{s_n}\right) - f\left(Z_k + \frac{Y_k}{s_n}\right) &= f'(Z_k)\left(\frac{X_n}{s_n} - \frac{Y_n}{s_n}\right) + \frac{f''(M_k)X_k^2}{2s_n^2} - \frac{f''(N_k)Y_k^2}{2s_n^2} \\
&= f'(Z_k)\left(\frac{X_n}{s_n} - \frac{Y_n}{s_n}\right) + \frac{1}{2}f''(Z_k)\left(\frac{X_k^2}{s_n^2} - \frac{Y_k^2}{s_n^2}\right) \\
&\quad + \frac{f''(M_k) - f''(Z_k)}{2}\frac{X_k^2}{s_n^2} - \frac{f''(N_k) - f''(Z_k)}{2}\frac{Y_k^2}{s_n^2} \\
&= I_1 + I_2 + I_3 - I_4.
\end{aligned}$$

By independence of the random variables X_k, Y_k, Z_k, and $\mathbb{E}\{X\} = \mathbb{E}\{Y\} = 0$, $\mathbb{E}\left\{X_k^2\right\} = \mathbb{E}\left\{Y_k^2\right\} = \sigma_k^2$, the expectation of $I_1 + I_2$ is null and the remainder is $I_3 - I_4$. Since $f \in \mathcal{C}_K^2$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|z - s| < \delta$ then $|f''(z) - f''(s)| < \varepsilon$. It follows that

$$|I_3 - I_4| \leq \frac{1}{s_n^2}\|f''\|_\infty\left(X_k^2 \mathbb{1}_{\{|X_k| > \delta s_n\}} + Y_k^2 \mathbb{1}_{\{|Y_k| > \delta s_n\}}\right) + \frac{\varepsilon}{2s_n^2}\left(X_k^2 + Y_k^2\right).$$

Thus, the upper bound of $\mathbb{E}\left\{f\left(Z_k + \frac{X_k}{s_n}\right) - f\left(Z_k + \frac{Y_k}{s_n}\right)\right\}$ is

$$\frac{1}{s_n^2}\|f''\|_\infty\left(\mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k| > \delta s_n\}}\right\} + \mathbb{E}\left\{Y_k^2 \mathbb{1}_{\{|Y_k| > \delta s_n\}}\right\}\right) + \frac{\varepsilon}{2s_n^2}\left(\mathbb{E}\left\{X_k^2\right\} + \mathbb{E}\left\{Y_k^2\right\}\right).$$

Taking the sum on $k = 1, 2, \ldots, n$, we have

$$|I_n| \leq \|f''\|_\infty \frac{1}{s_n^2}\sum_{k=1}^n\left(\mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k| > \delta s_n\}}\right\} + \mathbb{E}\left\{Y_k^2 \mathbb{1}_{\{|Y_k| > \delta s_n\}}\right\}\right) + \varepsilon.$$

For $n \to \infty$, the proof of this theorem will be completed if we show that $(Y_n)_{n\geq 1}$ also satisfies Lindeberg's condition (2.5).

For any $k = 1, 2, \ldots, n$, we have

$$\mathbb{E}\left\{Y_k^2 \mathbb{1}_{\{|Y_k| > \delta s_n\}}\right\} = \sigma_k^2 \mathbb{E}\left\{Y^2 \mathbb{1}_{\left\{|Y| > \frac{\delta s_n}{\sigma_k}\right\}}\right\} \leq \sigma_k^2 \mathbb{E}\left\{Y^2 \mathbb{1}_{\left\{|Y| > \frac{\delta s_n}{\sigma_j}\right\}}\right\},$$

where $\sigma_j = \max_{k \leq n}\{\sigma_k\}$ and $Y \sim \mathcal{N}(0, 1)$.
Again, taking the sum on $k = 1, 2, \ldots, n$, we have

$$\frac{1}{s_n^2}\sum_{k=1}^n \mathbb{E}\left\{Y_k^2 \mathbb{1}_{\{|Y_k| > \delta s_n\}}\right\} \leq \mathbb{E}\left\{Y^2 \mathbb{1}_{\left\{|Y| > \frac{\delta s_n}{\sigma_j}\right\}}\right\}. \tag{2.6}$$

And, the last one, since

$$\begin{aligned}
\frac{1}{s_n^2}\sum_{k=1}^n \mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k| > \delta s_n\}}\right\} &\geq \frac{1}{s_n^2}\mathbb{E}\left\{X_j^2 \mathbb{1}_{\{|X_j| > \delta s_n\}}\right\} \geq \frac{1}{s_n^2}\left(\sigma_j^2 - \mathbb{E}\left\{X_j^2 \mathbb{1}_{\{|X_j| \leq \delta s_n\}}\right\}\right) \\
&\geq \frac{1}{s_n^2}\left(\sigma_j^2 - \delta^2 s_n^2\right) = \left(\frac{\sigma_j}{s_n}\right)^2 - \delta^2,
\end{aligned}$$

by Lindeberg 's condition (2.5) of $(X_n)_{n\geq 1}$, for $n \to \infty$, we have $\lim_{n\to\infty} sup(\frac{\sigma_j}{s_n})^2 \leq \delta^2$ for any $\delta > 0$. Thus, $\lim_{n\to\infty} \frac{\sigma_j}{s_n} = 0$. Finally, in (2.6), for $n \to \infty$ we have

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{n} \mathbb{E}\left\{Y_k^2 \mathbb{1}_{\{|Y_k|>\delta s_n\}}\right\} = 0.$$

□

2.3 Central limit theorem for martingales

We begin with Billingsley's theorem for stationary martingale.

2.3.1 Stationary Martingale Central Limit theorem

Theorem 2.3.1. *(Billingsley, 1961) Suppose the sequence of $(X_n)_{n\geq 1}$ be stationary and ergodic such that $Var\{X_1\} = \mathbb{E}\{X_1^2\}$ is finite and*

$$\mathbb{E}\{X_n/X_1, \ldots, X_{n-1}\} = 0, \quad a.s. \tag{2.7}$$

Then the distribution of $\dfrac{1}{\sqrt{n}} \displaystyle\sum_{k=1}^{n} X_k$ tends to the normal distribution with mean 0 and variance $\mathbb{E}\{X_1^2\}$.

Proof. To prove this theorem, we may assume the process is represented in the way of Billingsley [4]. Let Ω be the Cartesian product of a sequence of copies of the real line, indexed by the integers $n = 0, \pm 1, \pm 2, \ldots$. Let X_n be the coordinate variables, let $\mathcal{B}$ be the Borel field generated by them, and let $\mathbb{P}$ be the probability measure on $\mathcal{B}$ with the finite dimensional distributions prescribed by the original process. Let $\mathcal{F}_n = \sigma(X_n, X_{n-1}, \ldots)$ then by (2.7)

$$\mathbb{E}\{X_n/\mathcal{F}_{n-1}\} = 0, \quad a.s. \tag{2.8}$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Let $\sigma_n^2 = \mathbb{E}\{X_n^2/\mathcal{F}_{n-1}\}$ and let $\sigma^2 = \mathbb{E}\{\sigma_n^2\} = \mathbb{E}\{X_n^2\}$. If T is the shift operator then $\sigma_n^2 = T^n \sigma_0^2$. Since the hypothesis of the sequence of $(X_n)_{n\geq 1}$ then T is ergodic, it follows by the ergodic theorem that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sigma_k^2 = \sigma^2, \quad a.s. \tag{2.9}$$

Let $q_n^2 = \sigma_1^2 + \ldots + \sigma_n^2$, put $m_t = \min\{n : q_n^2 \geq t\}$ for $t > 0$, let c_t be the number such that $0 < c_t \leq 1$ and $q_{m_t-1}^2 + c_t \sigma_{m_t}^2 = t$, and finally, let $Z_t = X_1 + \ldots + X_{m_t-1} + c_t X_{m_t}$. We see that m_t is well defined and so other variables by the following lemma

Lemma 2.3.1. *For $t > 0$ and m_t defined as above, then* *i) If* $t < \infty$ *then* $m_t < \infty$.

ii) $\lim\limits_{t\to\infty} m_t = \infty$.

Proof. For $t < \infty$, suppose that $m_t = \infty$. Then we have $m_t - 1 = \infty$, (2.9) shows that $q^2_{m_t-1} = \infty$. It follows that $q^2_{m_t-1} > t$. This is a contradiction, hence, $m_t < \infty$. This proved i).

For $t \to \infty$, suppose $m_t < N < \infty$ then $s^2_N < \infty$. By (2.9), $\sup\limits_n \frac{q_n^2}{n} < \infty$. Hence, $q^2_N < \infty$ and so $q^2_{m_t} < q^2_N < t$. This is a contradiction, hence, $m_t = \infty$. This proved ii). □

Furthermore, we have the second lemma for m_t.

Lemma 2.3.2. *As above,* $m_t = \min\{n : q^2_n \geq t\}$ *for* $t > 0$*, then we have*

$$\lim_{t\to\infty} \frac{t}{m_t} = \sigma^2. \tag{2.10}$$

Proof. For any $\delta > 0$, we have

$$\frac{q_{m_t}}{m_t} - \frac{\sigma^2_{m_t}}{m_t} \leq \frac{q_{m_t-1}}{m_t} \leq \frac{t}{m_t} \leq \frac{q_{m_t}}{m_t}.$$

By (2.9) and lemma 2.3.1, this lemma would be proved if we can show that

$$\lim_{n\to\infty} \frac{\sigma^2_n}{n} = 0. \tag{2.11}$$

Applying lemma 3.4.1, then for any $\varepsilon > 0$

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{\frac{\sigma^2_n}{n} > \varepsilon\right\} \;=\; \sum_{n=1}^{\infty} \mathbb{P}\left\{\frac{\sigma^2_n}{\varepsilon} > n\right\} \leq \frac{1}{\varepsilon}\mathbb{E}\left\{X_1^2\right\} = \frac{\sigma^2}{\varepsilon}.$$

By Borel Cantelli 's lemma, one has $\frac{\sigma^2_n}{n} < \varepsilon$ a.s for n large enough which completes (2.11). □

About Z_t, it will plays an important part in our proof because we in the sequel can show that

$$\frac{1}{\sqrt{t}} Z_t \xrightarrow{D} \mathcal{N}(0,1) \text{ as } t \to \infty. \tag{2.12}$$

And hence, the proof of the theorem will then be completed by showing that

$$\lim_{n\to+\infty} \mathbb{P}\left\{n^{-1/2}\left|\sum_{k=1}^{n} X_k - Z_{n\sigma^2}\right| > \varepsilon\right\} = 0, \quad \forall\varepsilon > 0. \tag{2.13}$$

To prove (2.13), we will use (2.10) and Kolmogorov's inequality for martingales. This is adapted from Billingsley (1961). From (2.10), given $\varepsilon > 0$, choose n_0 such that if $n \geq n_0$ then

$$\mathbb{P}\left\{\left|\frac{m_{n\sigma^2}}{n\sigma^2} - \sigma^{-2}\right| > \varepsilon^3\right\} < \varepsilon$$

it follows that
$$\mathbb{P}\left\{|m_{n\sigma^2} - n| > \varepsilon^3 n\sigma^2\right\} < \varepsilon.$$
If $n \geq n_0$ then
$$\mathbb{P}\left\{n^{-1/2}\left|\sum_{k=1}^{n} X_k - Z_{n\sigma^2}\right| > \varepsilon\right\} \leq \varepsilon + \mathbb{P}\left\{\max_{a\leq \ell\leq b}\left|\sum_{k=a}^{\ell} X_k\right| \geq \frac{\varepsilon n^{1/2}}{2}\right\}$$
where $a = n - \left[\varepsilon^3 n\sigma^2\right]$ and $b = n + \left[\varepsilon^3 n\sigma^2\right]$. By Kolmogorov's inequality for martingales
$$\mathbb{P}\left\{\max_{a\leq \ell\leq b}\left|\sum_{k=a}^{\ell} X_k\right| \geq \frac{\varepsilon n^{1/2}}{2}\right\} \leq \frac{4}{\varepsilon^2 n}\sum_{k=a}^{b}\mathbb{E}\left\{X_k^2\right\} \leq 8\varepsilon\sigma^2.$$
We have thus proved that
$$\mathbb{P}\left\{n^{-1/2}\left|\sum_{k=1}^{n} X_k - Z_{n\sigma^2}\right| > \varepsilon\right\} \leq (1+8\sigma^2)\varepsilon$$
if $n \geq n_0$, and we finish the proof of (2.13).

The remainder is to prove (2.12), we define new variables by
$$\tilde{X}_k = X_k \mathbb{1}_{\{m_t > k\}} + X_k c_t \mathbb{1}_{\{m_t = k\}}. \tag{2.14}$$
For $m_t > k$, that means $q_k^2 = \sum_{i=1}^{k}\sigma_i^2 < t$ implies $\{m_t > k\}$ is $\mathcal{F}_{k-1}$-measurable. Similarly, $\{m_t > k-1\}$ is $\mathcal{F}_{k-1}$-measurable and $\{m_t \leq k\}$ is the complement of $\{m_t > k\}$ is also $\mathcal{F}_{k-1}$-measurable; it follows that $\{m_t = k\} = \{m_t > k-1\} \cap \{m_t \leq k\}$ is $\mathcal{F}_{k-1}$-measurable, and hence $c_t \mathbb{1}_{\{m_t=k\}}$ is $\mathcal{F}_{k-1}$-measurable since $c_t = \frac{t-q_{k-1}^2}{\sigma_k^2}$ on $\{m_t = k\}$. Therefore, if $\tilde{\sigma}_k^2 = \mathbb{E}\left\{\tilde{X}_k^2/\mathcal{F}_{k-1}\right\}$, then we have
$$\tilde{\sigma}_k^2 = \sigma_k^2 \mathbb{1}_{\{m_t>k\}} + c_t^2\sigma_k^2\mathbb{1}_{\{m_t=k\}} \tag{2.15}$$
and so
$$\sum_{k=1}^{\infty}\tilde{\sigma}_k^2 = t \tag{2.16}$$
except on a set of measure zero. Moreover, we also have $\mathbb{E}\left\{\tilde{X}_k/\mathcal{F}_{k-1}\right\} = 0, a.s.$

Adjoin to the space random variables $\xi_1, \xi_2, \ldots$, each normally distributed with mean 0 and variance 1, which are independent of each other and of the Borel field $\mathcal{B}$. If we put new variables
$$\eta_n = \frac{1}{\sqrt{t}}\left(\tilde{X}_1 + \tilde{X}_2 + \ldots + \tilde{X}_n + \tilde{\sigma}_{n+1}\xi_{n+1} + \tilde{\sigma}_{n+2}\xi_{n+2} + \ldots\right)$$
then $\eta_n = \dfrac{1}{\sqrt{t}}Z_t$, for any $n \geq m_t$. Moreover, since $\mathbb{E}\left\{\eta_0/\mathcal{B}\right\} = 0$, $\mathbb{E}\left\{\eta_0^2/\mathcal{B}\right\} = \dfrac{1}{t}\displaystyle\sum_{k=1}^{\infty}\tilde{\sigma}_k^2 = 1$, then η_0 has the standard normal distribution because of the independence of $(\xi_i)_{i\geq 1}$. For any function $f \in \mathcal{C}_k^2$ we set

$$I_t(f) = \mathbb{E}\left\{f\left(\frac{1}{\sqrt{t}}Z_t\right) - f(\eta_0)\right\} = \sum_{k=1}^{\infty} \mathbb{E}\left\{f(\eta_k) - f(\eta_{k-1})\right\}. \tag{2.17}$$

If we put

$$W_n = \frac{1}{\sqrt{t}}\left(\tilde{X}_1 + \tilde{X}_2 + \ldots + \tilde{X}_{n-1} + \tilde{\sigma}_{n+1}\xi_{n+1} + \tilde{\sigma}_{n+2}\xi_{n+2} + \ldots\right)$$

then by Taylor's expansions, we have

$$\begin{aligned} f(\eta_k) &= f(W_k) + f'(W_k)t^{-1/2}\tilde{X}_k + \frac{1}{2}f''(M_k)t^{-1}\tilde{X}_k^2 \\ f(\eta_{k-1}) &= f(W_k) + f'(W_k)t^{-1/2}\tilde{\sigma}_k\xi_k + \frac{1}{2}f''(N_k)t^{-1}\tilde{\sigma}_k^2\xi_k^2 \end{aligned}$$

for some M_k and N_k such that $|M_k - W_k| \leq |\tilde{X}_k|/\sqrt{t}$ and $|N_k - W_k| \leq |\tilde{\sigma}_k\xi_k|/\sqrt{t}$. Therefore we have the following calculation

$$\begin{aligned} f(\eta_k) - f(\eta_{k-1}) &= f'(W_k)\frac{1}{\sqrt{t}}\left(\tilde{X}_k - \tilde{\sigma}_k\xi_k\right) + \frac{1}{2t}f''(M_k)\tilde{X}_k^2 + \frac{1}{2t}f''(N_k)\tilde{\sigma}_k^2\xi_k^2 \\ &= f'(W_k)\frac{1}{\sqrt{t}}\left(\tilde{X}_k - \tilde{\sigma}_k\xi_k\right) + \frac{1}{2t}f''(W_k)\left(\tilde{X}_k^2 - \tilde{\sigma}_k^2\xi_k^2\right) \\ &\quad + \frac{1}{2t}\left(f''(M_k) - f''(W_k)\right)\tilde{X}_k^2 - \frac{1}{2t}\left(f''(N_k) - f''(W_k)\right)\tilde{\sigma}_k^2\xi_k^2 \\ &= I_1 + I_2 + I_3 - I_4. \end{aligned} \tag{2.18}$$

We have

$$\begin{aligned} \mathbb{E}\{I_1\} &= \mathbb{E}\{\mathbb{E}\{I_1/\xi_k, \mathcal{B}\}\} \\ &= \mathbb{E}\left\{\frac{1}{\sqrt{t}}\left(\tilde{X}_k - \tilde{\sigma}_k\xi_k\right)\mathbb{E}\left\{f'(W_k)/\xi_k, \mathcal{B}\right\}\right\}. \end{aligned}$$

If we put

$$W_k^1 = \frac{1}{\sqrt{t}}\left(\tilde{X}_1 + \tilde{X}_2 + \ldots + \tilde{X}_{k-1}\right)$$

and

$$W_k^2 = \frac{1}{\sqrt{t}}\left(\tilde{\sigma}_{k+1}\xi_{k+1} + \tilde{\sigma}_{k+2}\xi_{k+2} + \ldots\right)$$

then

$$f'(W_k) = f'\left(W_k^1 + W_k^2\right)$$

and the law of W_k^1 knowing $\{\xi_k, \mathcal{B}\}$ is $\mathcal{F}_{k-1}-$measurable, the law of W_k^2 knowing $\{\xi_k, \mathcal{B}\}$ $\sim \mathcal{N}\left(0, \frac{1}{t}\sum_{\ell=k+1}^{\infty}\tilde{\sigma}_\ell^2\right) \sim \mathcal{N}\left(0, 1 - \frac{1}{t}\sum_{\ell=1}^{k}\tilde{\sigma}_\ell^2\right)$ is also $\mathcal{F}_{k-1}-$measurable. They follow that $\mathbb{E}\left\{f'(W_k)/\xi_k, \mathcal{B}\right\}$ is $\mathcal{F}_{k-1}-$measurable. And hence

$$\mathbb{E}\{I_1\} = \mathbb{E}\left\{\frac{1}{\sqrt{t}}\mathbb{E}\left\{\left(\tilde{X}_k - \tilde{\sigma}_k\xi_k\right)\mathbb{E}\left\{f'(W_k)/\xi_k, \mathcal{B}\right\}/\mathcal{F}_{k-1}\right\}\right\}$$

$$\begin{aligned}
&= \mathbb{E}\left\{\frac{1}{\sqrt{t}}\mathbb{E}\left\{f'(W_k)/\xi_k,\mathcal{B}\right\}\mathbb{E}\left\{\left(\tilde{X}_k-\tilde{\sigma}_k\xi_k\right)/\mathcal{F}_{k-1}\right\}\right\}\\
&= 0.
\end{aligned}$$

Similarly, we have also

$$\begin{aligned}
\mathbb{E}\{I_2\} &= \mathbb{E}\left\{\frac{1}{2t}\mathbb{E}\left\{f''(W_k)\left(\tilde{X}_k^2-\tilde{\sigma}_k^2\xi_k^2\right)/\xi_k,\mathcal{B}\right\}\right\}\\
&= \mathbb{E}\left\{\frac{1}{2t}\left(\tilde{X}_k^2-\tilde{\sigma}_k^2\xi_k^2\right)\mathbb{E}\left\{f''(W_k)/\xi_k,\mathcal{B}\right\}\right\}\\
&= \mathbb{E}\left\{\frac{1}{2t}\mathbb{E}\left\{\left(\tilde{X}_k^2-\tilde{\sigma}_k^2\xi_k^2\right)\mathbb{E}\left\{f''(W_k)/\xi_k,\mathcal{B}\right\}/\mathcal{F}_{k-1}\right\}\right\}\\
&= \mathbb{E}\left\{\frac{1}{2t}\mathbb{E}\left\{f''(W_k)/\xi_k,\mathcal{B}\right\}\mathbb{E}\left\{\left(\tilde{X}_k^2-\tilde{\sigma}_k^2\xi_k^2\right)/\mathcal{F}_{k-1}\right\}\right\}\\
&= 0.
\end{aligned}$$

And the remainder is $I_3 - I_4$. Since $f \in \mathcal{C}_K^2$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|z-s| < \delta$ then $|f''(z) - f''(s)| < \varepsilon$. It follows that

$$|I_3 - I_4| \le \|f''\|_\infty \frac{1}{t}\tilde{X}_k^2 \mathbb{1}_{\{|\tilde{X}_k|>\delta\sqrt{t}\}} + \tilde{\sigma}_k^2\xi_k^2\mathbb{1}_{\{|\tilde{\sigma}_k\xi_k|>\delta\sqrt{t}\}} + \frac{\varepsilon}{2t}\left(\tilde{X}_k^2+\tilde{\sigma}_k^2\xi_k^2\right).$$

Thus, the upper bound of $\mathbb{E}\{(f(\eta_k) - f(\eta_{k-1}))\}$ is

$$\begin{aligned}
&\|f''\|_\infty\mathbb{E}\left\{\frac{1}{t}\mathbb{E}\left\{\tilde{X}_k^2\mathbb{1}_{\{|\tilde{X}_k|>\delta\sqrt{t}\}}/\mathcal{F}_{k-1}\right\} + \mathbb{E}\left\{\tilde{\sigma}_k^2\xi_k^2\mathbb{1}_{\{|\tilde{\sigma}_k\xi_k|>\delta\sqrt{t}\}}\right\}\right\}\\
&\qquad\qquad +\varepsilon\mathbb{E}\left\{\frac{1}{2t}\mathbb{E}\left\{\tilde{X}_k^2/\mathcal{F}_{k-1}\right\}+\mathbb{E}\left\{\tilde{\sigma}_k^2\xi_k^2\right\}\right\}.
\end{aligned}$$

Taking the sum on $k = 1, 2, \ldots$, we have

$$\begin{aligned}
|I_t(f)| &= \left|\mathbb{E}\left\{f\left(\frac{1}{\sqrt{t}}Z_t\right) - f(\eta_0)\right\}\right|\\
&\le \|f''\|_\infty\mathbb{E}\left\{\frac{1}{t}\sum_{k=1}^{\infty}\left[\mathbb{E}\left\{\tilde{X}_k^2\mathbb{1}_{\{|\tilde{X}_k|>\delta\sqrt{t}\}}/\mathcal{F}_{k-1}\right\}+\mathbb{E}\left\{\tilde{\sigma}_k^2\xi_k^2\mathbb{1}_{\{|\tilde{\sigma}_k\xi_k|>\delta\sqrt{t}\}}\right\}\right]\right\}\\
&\quad +\frac{\varepsilon}{2t}\mathbb{E}\left\{\sum_{k=1}^{\infty}\left[\mathbb{E}\left\{\tilde{X}_k^2/\mathcal{F}_{k-1}\right\}+\tilde{\sigma}_k^2\xi_k^2\right]\right\}\\
&\le \|f''\|_\infty\mathbb{E}\left\{\frac{1}{t}\sum_{k=1}^{m_t}\left[\mathbb{E}\left\{\tilde{X}_k^2\mathbb{1}_{\{|\tilde{X}_k|>\delta\sqrt{t}\}}/\mathcal{F}_{k-1}\right\}+\mathbb{E}\left\{\tilde{\sigma}_k^2\xi_k^2\mathbb{1}_{\{|\tilde{\sigma}_k\xi_k|>\delta\sqrt{t}\}}\right\}\right]\right\}\\
&\quad +\frac{\varepsilon}{2t}\mathbb{E}\left\{\sum_{k=1}^{m_t}\left[\mathbb{E}\left\{\tilde{X}_k^2/\mathcal{F}_{k-1}\right\}+\tilde{\sigma}_k^2\xi_k^2\right]\right\}\\
&\le \|f''\|_\infty\mathbb{E}\left\{\frac{1}{t}\sum_{k=1}^{m_t}\left[\mathbb{E}\left\{\tilde{X}_k^2\mathbb{1}_{\{|\tilde{X}_k|>\delta\sqrt{t}\}}/\mathcal{F}_{k-1}\right\}+\mathbb{E}\left\{\tilde{\sigma}_k^2\xi_1^2\mathbb{1}_{\{|\tilde{\sigma}_k\xi_1|>\delta\sqrt{t}\}}\right\}\right]\right\}\\
&\quad +\varepsilon.
\end{aligned}$$

Since $\sum_{k=1}^{m_t} \tilde{\sigma}_k^2 = t$, the integrand in last expression is bounded by 2. Therefore, for $t \to \infty$, by the dominated convergence theorem, the proof of (2.12) will be completed if we show that

$$\frac{1}{t}\sum_{k=1}^{m_t} \mathbb{E}\left\{\tilde{X}_k^2 \mathbb{1}_{\{|\tilde{X}_k|>\delta\sqrt{t}\}}/\mathcal{F}_{k-1}\right\} \longrightarrow 0 \tag{2.19}$$

and

$$\frac{1}{t}\sum_{k=1}^{m_t} \mathbb{E}\left\{\tilde{\sigma}_k^2 \xi_1^2 \mathbb{1}_{\{|\tilde{\sigma}_k\xi_1|>\delta\sqrt{t}\}}\right\} \longrightarrow 0 \tag{2.20}$$

by using ergodicity, stationarity of $(X_n)_{n\geq 1}$ and lemma 2.3.2.

Proof of (2.19). For $u > 0$, then $t > u$ for t large enough. And, for any $k = 1, 2, \ldots, n$, we have

$$\limsup_{t\to\infty} \frac{1}{t}\sum_{k=1}^{m_t} \mathbb{E}\left\{\tilde{X}_k^2 \mathbb{1}_{\{|\tilde{X}_k|>\delta\sqrt{t}\}}/\mathcal{F}_{k-1}\right\} \leq \limsup_{t\to\infty} \frac{1}{t}\sum_{k=1}^{m_t} \mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k|>\delta\sqrt{u}\}}/\mathcal{F}_{k-1}\right\}. \tag{2.21}$$

If follows from the ergodic theorem that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k|>\delta\sqrt{u}\}}/\mathcal{F}_{k-1}\right\} = \mathbb{E}\left\{X_1^2 \mathbb{1}_{\{|X_1|>\delta\sqrt{u}\}}\right\}. \tag{2.22}$$

From (2.10) and (2.22), we have that the left hand member of (2.21) is bounded by

$$\sigma^{-2}\mathbb{E}\left\{X_1^2 \mathbb{1}_{\{|X_1|>\delta\sqrt{u}\}}\right\}$$

For $u \to \infty$, this bound goes to 0 by the dominated convergence theorem, the left hand member of (2.21) is 0 a.s.

Proof of (2.20). Similarly, for $v > 0$, then $t > v$ for t large enough. And, for any $k = 1, 2, \ldots, n$, we have

$$\limsup_{t\to\infty} \frac{1}{t}\sum_{k=1}^{m_t} \mathbb{E}\left\{\tilde{\sigma}_k^2 \xi_1^2 \mathbb{1}_{\{|\tilde{\sigma}_k\xi_1|>\delta\sqrt{v}\}}\right\} \leq \limsup_{t\to\infty} \frac{1}{t}\sum_{k=1}^{m_t} \mathbb{E}\left\{\sigma_k^2 \xi_1^2 \mathbb{1}_{\{|\sigma_k\xi_1|>\delta\sqrt{v}\}}\right\}. \tag{2.23}$$

If follows from the ergodic theorem that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}\left\{\sigma_k^2 \xi_1^2 \mathbb{1}_{\{|\sigma_k\xi_1|>\delta\sqrt{v}\}}\right\} = \mathbb{E}\left\{\sigma_1^2 \xi_1^2 \mathbb{1}_{\{|\sigma_1\xi_1|>\delta\sqrt{v}\}}\right\}. \tag{2.24}$$

From (2.10) and (2.24), we have that the left hand member of (2.23) is bounded by

$$\sigma^{-2}\mathbb{E}\left\{\sigma_1^2 \xi_1^2 \mathbb{1}_{\{|\sigma_1\xi_1|>\delta\sqrt{v}\}}\right\}$$

For $v \to \infty$, this bound goes to 0 by the dominated convergence theorem, the left hand member of (2.23) is 0 a.s. Thus, the integrand on the right in (2.19) goes to 0 a.s., which completes the proof of (2.12).

□

2.3.2 Martingale Central Limit Theorem

Let $(X_n)_{n\geq 0}$ be a sequence of random variables defined over the probability space $(\Omega, \mathcal{A}, \mathbb{P})$, $\mathcal{B} = \sigma(X_n, n = 0,1,2,3,\ldots)$ and $\mathcal{F}_n = \sigma(X_n, X_{n-1},\ldots)$. Assume that the partial sums of X_n define a martingale: X_n is $\mathcal{F}_n$-measurable and $\mathbb{E}\left\{X_n/\mathcal{F}_{n-1}\right\} = 0$ for $n \geq 1$. Put $s_n^2 = \sum\limits_{k=1}^{n} \mathbb{E}\left\{X_k^2\right\}$.

Theorem 2.3.2. *Assume that the following limits hold almost surely (a.s.)*

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{n} \mathbb{E}\left\{X_k^2/\mathcal{F}_{k-1}\right\} = 1, \tag{2.25}$$

and

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{n} \mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k|>\delta s_n\}}/\mathcal{F}_{k-1}\right\} = 0 \tag{2.26}$$

then the distribution of $\frac{1}{s_n}\sum\limits_{k=1}^{n} X_k$ *tends to the standard normal law* $\mathcal{N}(0,1)$ *when* $n \to \infty$.

***Remark* 2.3.1.** *In 1971, in [8] Brown proved Theorem 2.3.2 where conditions (2.25), (2.26) hold in probability but we use with almost surely convergence.*

Before proving Theorem 2.3.2, we need two lemmas as follows

Lemma 2.3.3. *The conditions (2.25), (2.26) in Theorem 2.3.2 hold in* L^1.

Proof. Put

$$G_n = \frac{1}{s_n^2} \sum_{k=1}^{n} \mathbb{E}\left\{X_k^2/\mathcal{F}_{k-1}\right\} - 1.$$

By (2.25), $G_n \to 0$ a.s. Decompositing $G_n = G_n^+ - G_n^-$. Since, $G_n \geq -1$ for any n then $0 \leq G_n^- \leq 1$ and follows that $\mathbb{E}\left\{G_n^-\right\} \to 0$ by the dominated convergence theorem. Moreover, $\mathbb{E}\left\{G_n\right\} = 0$, implies $\mathbb{E}\left\{G_n^+\right\} = \mathbb{E}\left\{G_n^-\right\}$, for any $n \geq 1$, then

$$\lim_{n\to\infty} \mathbb{E}\left\{G_n^+\right\} = \lim_{n\to\infty} \mathbb{E}\left\{G_n^-\right\} = 0.$$

Therefore

$$\lim_{n\to\infty} \mathbb{E}\left\{|G_n|\right\} \leq \lim_{n\to\infty} \mathbb{E}\left\{G_n^+ + G_n^-\right\} = 0$$

and hence $G_n \longrightarrow 0$ in L^1. In the other hand, by putting

$$\begin{aligned} H_n &= \frac{\sum\limits_{k=1}^{n} \mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k|>\delta s_n\}}/\mathcal{F}_{k-1}\right\}}{\sum\limits_{k=1}^{n} \mathbb{E}\left\{X_k^2/\mathcal{F}_{k-1}\right\}}, \\ K_n &= \frac{1}{s_n^2} \sum_{k=1}^{n} \mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k|>\delta s_n\}}/\mathcal{F}_{k-1}\right\} \end{aligned}$$

we obtain $K_n = (1 + G_n)H_n$ for any n. By condition (2.26),

$$\lim_{n\to\infty} K_n = \lim_{n\to\infty} (1 + G_n)H_n = 0 \text{ a.s.}$$

and since $G_n \to 0$ a.s. by (2.25), we deduce that $H_n \to 0$ a.s.

Moreover $0 \le H_n \le 1$, by the dominated convergence theorem then follows that $\mathbb{E}\{H_n\} \to 0$ as $n \to \infty$. Finally,

$$\mathbb{E}\{K_n\} = \mathbb{E}\{H_n\} + \mathbb{E}\{G_n H_n\} \le \mathbb{E}\{H_n\} + \mathbb{E}\{G_n\} \longrightarrow 0 \text{ as } n \to \infty.$$

Hence, $K_n \to 0$ in L^1. □

Lemma 2.3.4. $\lim_{n\to\infty} s_n^2 = \sum_{k=1}^{\infty} \mathbb{E}\{X_k^2\} = \infty.$

Proof. Assume $\lim_{n\to\infty} s_n^2 = M^2 < \infty$, then there exists $N > 0$ such that

$$s_N^2 > M^2 - M^2/3. \tag{2.27}$$

By (2.26)

$$\sum_{k=1}^{\infty} \mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k| > \delta M\}} / \mathcal{F}_{k-1}\right\} = 0, \quad a.s.$$

This implies $\mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k| > \delta M\}} / \mathcal{F}_{k-1}\right\} = 0$ a.s. and follows that $\mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k| > \delta M\}}\right\} = 0$ for any $k \ge 1$ and for any $\delta > 0$. Therefore,

$$s_N^2 = \sum_{k=1}^{N} \mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k| \le \delta M\}}\right\} \le N\delta^2 M^2. \tag{2.28}$$

Choose $\delta = 1/2\sqrt{N}$, (2.27) and (2.28) give a contradiction ! □

Proof of Theorem 2.3.2. Let $\sigma_k^2 = \mathbb{E}\left\{X_k^2/\mathcal{F}_{k-1}\right\}$ and let $q_n^2 = \sigma_1^2 + \ldots + \sigma_n^2$, put $m_t = \min\{n : q_n^2 \ge t\}$ for $t > 0$, let c_t be the number such that $0 < c_t \le 1$ and $q_{m_t-1}^2 + c_t \sigma_{m_t}^2 = t$, and finally, let $Z_t = X_1 + \ldots + X_{m_t-1} + c_t X_{m_t}$. We see that m_t is well defined and so other variables by the following lemma

Lemma 2.3.5. *For $t > 0$ and m_t defined as above, then*

i) If $t < \infty$ then $m_t < \infty$.

ii) $\lim_{t\to\infty} m_t = \infty$.

Proof. For $t < \infty$, suppose that $m_t = \infty$. By lemma 2.3.4 we have $s_{m_t-1}^2 = \infty$, (2.25) implies $q_{m_t-1}^2 = \infty$. It follows that $q_{m_t-1}^2 > t$. This is a contradiction, hence, $m_t < \infty$. This proved *i*).

For $t \to \infty$, suppose $m_t < N < \infty$ then $s_N^2 < \infty$. By (2.25), $\lim_{n\to\infty} \frac{q_n^2}{s_n^2} = 1$ implies $\sup_n \frac{q_n^2}{s_n^2} < \infty$. Hence, $q_N^2 < \infty$ and so $q_{m_t}^2 < q_N^2 < t$. This is a contradiction, hence, $m_t = \infty$. This proved *ii*). □

Furthermore, we have the second lemma for m_t

Lemma 2.3.6. *We recall $m_t = \min\{n : q_n^2 \geq t\}$ for $t > 0$ then we have*

$$\lim_{t\to\infty} \frac{t}{s_{m_t}^2} = 1. \tag{2.29}$$

Proof. We have

$$\frac{q_{m_t}^2}{s_{m_t}^2} - \frac{\sigma_{m_t}^2}{s_{m_t}^2} = \frac{q_{m_t}^2 - \sigma_{m_t}^2}{s_{m_t}^2} \leq \frac{q_{m_t-1}^2}{s_{m_t}^2} \leq \left(\frac{t}{s_{m_t}^2}\right) \leq \frac{q_{m_t}^2}{s_{m_t}^2}.$$

By (2.25) and lemma 2.3.5, this lemma would be proved if we can show that

$$\lim_{n\to\infty} \frac{\sigma_n^2}{s_n^2} = 0. \tag{2.30}$$

For any $\delta > 0$, we have

$$\begin{aligned} \frac{1}{s_n^2}\sum_{k=1}^{n} \mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k|>\delta s_n\}}/\mathcal{F}_{k-1}\right\} &\geq \frac{1}{s_n^2}\mathbb{E}\left\{X_n^2 \mathbb{1}_{\{|X_n|>\delta s_n\}}/\mathcal{F}_{n-1}\right\} \\ &= \frac{\sigma_n^2}{s_n^2} - \frac{1}{s_n^2}\mathbb{E}\left\{X_n^2 \mathbb{1}_{\{|X_n|\leq\delta s_n\}}/\mathcal{F}_{n-1}\right\} \\ &\geq \frac{\sigma_n^2}{s_n^2} - \delta^2. \end{aligned}$$

By (2.26), for $n \to \infty$, we have $\limsup_{n\to\infty} \frac{\sigma_n^2}{s_n^2} \leq \delta^2$ for any $\delta > 0$. Thus, we obtain the desired result. $\square$

About rv's Z_t, it plays an important role in our proof because in the sequel we can show that

$$\frac{1}{\sqrt{t}} Z_t \xrightarrow{D} \mathcal{N}(0,1) \text{ as } t \to \infty. \tag{2.31}$$

And hence, the proof of the theorem will be completed by showing that

$$\lim_{n\to+\infty} \mathbb{P}\left\{\frac{1}{s_n}\left|\sum_{k=1}^{n} X_k - Z_{s_n^2}\right| > \varepsilon\right\} = 0, \quad \forall\varepsilon > 0. \tag{2.32}$$

To prove (2.32), we will use (2.29) and Kolmogorov's inequality for martingales. From (2.29), given $\varepsilon > 0$, choose n_0 such that if $n \geq n_0$ then

$$\mathbb{P}\left\{\left|\frac{s_{m_{s_n^2}}^2}{s_n^2} - 1\right| > \varepsilon^3\right\} < \varepsilon$$

that means

$$\mathbb{P}\left\{\left|s_{m_{s_n^2}}^2 - s_n^2\right| > \varepsilon^3 s_n^2\right\} < \varepsilon$$

If $n \geq n_0$ then

$$\mathbb{P}\left\{\frac{1}{s_n}\left|\sum_{k=1}^{n} X_k - Z_{s_n^2}\right| > \varepsilon\right\} \leq \mathbb{P}\left\{\max_{a\leq \ell\leq b}\left|\sum_{k=a}^{\ell} X_k\right| \geq \frac{\varepsilon s_n}{2}\right\}$$

where $a = n + m_{s_n^2} - \left|n - m_{s_n^2}\right|$ and $b = n + m_{s_n^2} + \left|n - m_{s_n^2}\right|$. By Kolmogorov's inequality for martingales

$$\mathbb{P}\left\{\max_{a\leq \ell\leq b}\left|\sum_{k=a}^{\ell} X_k\right| \geq \frac{\varepsilon s_n}{2}\right\} \leq \varepsilon + 4\frac{\varepsilon^3 s_n^2}{\varepsilon^2 s_n^2} \leq 5\varepsilon.$$

We have thus proved that

$$\mathbb{P}\left\{\frac{1}{s_n}\left|\sum_{k=1}^{n} X_k - Z_{s_n^2}\right| > \varepsilon\right\} \leq 5\varepsilon$$

for $n \geq n_0$, and we finish the proof of (2.32).

The remainder is to prove (2.31), we define new variables by

$$\tilde{X}_k = X_k \mathbb{1}_{\{m_t > k\}} + X_k c_t \mathbb{1}_{\{m_t = k\}} \tag{2.33}$$

Similar arguments as in preceeding section, if $\tilde{\sigma}_k^2 = \mathbb{E}\left\{\tilde{X}_k^2/\mathcal{F}_{k-1}\right\}$, then we have

$$\tilde{\sigma}_k^2 = \sigma_k^2 \mathbb{1}_{\{m_t > k\}} + c_t^2 \sigma_k^2 \mathbb{1}_{\{m_t = k\}} \tag{2.34}$$

and so

$$\sum_{k=1}^{\infty} \tilde{\sigma}_k^2 = t \tag{2.35}$$

except on a set of measure zero. Moreover, we also have $\mathbb{E}\left\{\tilde{X}_k/\mathcal{F}_{k-1}\right\} = 0, a.s.$

Adjoin to the space random variables $\xi_1, \xi_2, \ldots$, each normally distributed with mean 0 and variance 1, which are independent of each other and of the Borel field $\mathcal{B}$. If we put new variables

$$\eta_n = \frac{1}{\sqrt{t}}\left(\tilde{X}_1 + \tilde{X}_2 + \ldots + \tilde{X}_n + \tilde{\sigma}_{n+1}\xi_{n+1} + \tilde{\sigma}_{n+2}\xi_{n+2} + \ldots\right)$$

then $\eta_n = \frac{1}{\sqrt{t}} Z_t$, for $n \geq m_t$. Moreover, since $\mathbb{E}\left\{\eta_0/\mathcal{B}\right\} = 0$, $\mathbb{E}\left\{\eta_0^2/\mathcal{B}\right\} = \frac{1}{t}\sum_{k=1}^{\infty}\tilde{\sigma}_k^2 = 1$, then η_0 has the standard normal distribution.

Write

$$W_n = \frac{1}{\sqrt{t}}\left(\tilde{X}_1 + \tilde{X}_2 + \ldots + \tilde{X}_{n-1} + \tilde{\sigma}_{n+1}\xi_{n+1} + \tilde{\sigma}_{n+2}\xi_{n+2} + \ldots\right).$$

For any function $f \in \mathcal{C}_k^2$, put

$$I_t(f) = \mathbb{E}\left\{f\left(\frac{1}{\sqrt{t}} Z_t\right) - f(\eta_0)\right\} = \sum_{k=1}^{\infty} \mathbb{E}\left\{f(\eta_k) - f(\eta_{k-1})\right\} \tag{2.36}$$

and then using the same method in the preceding section, we also have

$$\begin{aligned}
|I_t(f)| &= \left|\mathbb{E}\left\{f\left(\frac{1}{\sqrt{t}}Z_t\right)-f(\eta_0)\right\}\right| \\
&\leq \|f''\|_\infty \mathbb{E}\left\{\frac{1}{t}\sum_{k=1}^{\infty}\left[\mathbb{E}\left\{\tilde{X}_k^2 \mathbb{1}_{\{|\tilde{X}_k|>\delta\sqrt{t}\}}/\mathcal{F}_{k-1}\right\}+\mathbb{E}\left\{\tilde{\sigma}_k^2\xi_k^2\mathbb{1}_{\{|\tilde{\sigma}_k\xi_k|>\delta\sqrt{t}\}}\right\}\right]\right\} \\
&\quad +\frac{\varepsilon}{2t}\mathbb{E}\left\{\sum_{k=1}^{\infty}\left[\mathbb{E}\left\{\tilde{X}_k^2/\mathcal{F}_{k-1}\right\}+\tilde{\sigma}_k^2\xi_k^2\right]\right\} \\
&\leq \|f''\|_\infty \mathbb{E}\left\{\frac{1}{t}\sum_{k=1}^{m_t}\left[\mathbb{E}\left\{\tilde{X}_k^2 \mathbb{1}_{\{|\tilde{X}_k|>\delta\sqrt{t}\}}/\mathcal{F}_{k-1}\right\}+\mathbb{E}\left\{\tilde{\sigma}_k^2\xi_1^2\mathbb{1}_{\{|\tilde{\sigma}_k\xi_1|>\delta\sqrt{t}\}}\right\}\right]\right\} \\
&\quad +\varepsilon.
\end{aligned}$$

Since $\sum_{k=1}^{m_t}\tilde{\sigma}_k^2 = t$, the integrand in last expression is bounded by 2. Therefore, for $t \to \infty$, by the dominated convergence theorem, the proof of (2.31) will be completed if we show that

$$\frac{1}{t}\sum_{k=1}^{m_t}\mathbb{E}\left\{\tilde{X}_k^2 \mathbb{1}_{\{|\tilde{X}_k|>\delta\sqrt{t}\}}/\mathcal{F}_{k-1}\right\} \longrightarrow 0 \tag{2.37}$$

and

$$\frac{1}{t}\sum_{k=1}^{m_t}\mathbb{E}\left\{\tilde{\sigma}_k^2\xi_1^2\mathbb{1}_{\{|\tilde{\sigma}_k\xi_1|>\delta\sqrt{t}\}}\right\} \longrightarrow 0 \tag{2.38}$$

by using (2.25), (2.26) and lemma 2.3.6.

Proof of (2.37). We have

$$\begin{aligned}
\frac{1}{t}\sum_{k=1}^{m_t}\mathbb{E}\left\{\tilde{X}_k^2 \mathbb{1}_{\{|\tilde{X}_k|>\delta\sqrt{t}\}}/\mathcal{F}_{k-1}\right\} &\leq \frac{1}{t}\sum_{k=1}^{m_t}\mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k|>\delta s_{m_t-1}\}}/\mathcal{F}_{k-1}\right\} \\
&\leq \frac{s_{m_t-1}^2}{t}\cdot\frac{1}{s_{m_t-1}^2}\sum_{k=1}^{m_t-1}\mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k|>\delta s_{m_t-1}\}}/\mathcal{F}_{k-1}\right\} \\
&\quad +\frac{s_{m_t}^2}{t}\cdot\frac{\sigma_{m_t}^2}{s_{m_t}^2}.
\end{aligned}$$

By (2.29) and (2.26), we have

$$\lim_{t\to\infty}\frac{s_{m_t-1}^2}{t}\cdot\frac{1}{s_{m_t-1}^2}\sum_{k=1}^{m_t-1}\mathbb{E}\left\{X_k^2 \mathbb{1}_{\{|X_k|>\delta s_{m_t-1}\}}/\mathcal{F}_{k-1}\right\} = 0,$$

and

$$\lim_{t\to\infty}\frac{s_{m_t}^2}{t}\cdot\frac{\sigma_{m_t}^2}{s_{m_t}^2} = 0.$$

Hence, we obtain (2.37).

Proof of (2.38). Let $b_t = \max_{1\leq k\leq m_t}\left\{\frac{\sigma_k}{\sqrt{t}}\right\}$, we have $\lim_{t\to\infty}\frac{\sigma_k^2}{s_{m_t}^2} = 0$ for any $1 \leq k \leq m_t$. By (2.29), $\lim_{t\to\infty}\frac{\sigma_k}{\sqrt{t}} = \lim_{t\to\infty}\frac{\sigma_k}{s_{m_t}} = 0$ implies $\lim_{t\to\infty} b_t = 0$. Therefore

$$\begin{aligned}
\limsup_{t\to\infty} \frac{1}{t}\sum_{k=1}^{m_t} \mathbb{E}\left\{\tilde{\sigma}_k^2 \xi_1^2 \mathbb{1}_{\{|\tilde{\sigma}_k \xi_1| > \delta\sqrt{t}\}}\right\} &\leq \limsup_{t\to\infty} \frac{1}{t}\sum_{k=1}^{m_t} \mathbb{E}\left\{\sigma_k^2 \xi_1^2 \mathbb{1}_{\{|\sigma_k \xi_1| > \delta\sqrt{t}\}}\right\} \\
&\leq \limsup_{t\to\infty} \frac{1}{t}\sum_{k=1}^{m_t} \mathbb{E}\left\{\sigma_k^2 \xi_1^2 \mathbb{1}_{\{|\xi_1| > \delta b_t^{-1}\}}\right\} \\
&\leq \limsup_{t\to\infty} \frac{s_{m_t}^2}{t} \mathbb{E}\left\{\xi_1^2 \mathbb{1}_{\{|\xi_1| > \delta b_t^{-1}\}}\right\}. \qquad (2.39)
\end{aligned}$$

Since $\lim_{t\to\infty} b_t = 0$ implies $\lim_{t\to\infty} b_t^{-1} = +\infty$, then $\limsup_{t\to\infty} \mathbb{E}\left\{\xi_1^2 \mathbb{1}_{\{|\xi_1| > \delta b_t^{-1}\}}\right\} = 0$ which completes the proof of (2.38).

□

Chapter 3

Central limit theorem for Markov chain started at a point

This chapter is devoted to obtain CLT for Markov chain started at a point based on martingale approximation.

We begin with *Hopf Maximal Ergodic Theorem.*

3.1 Hopf Maximal Ergodic Theorem

We recall $(X_n)_{n\geq 0}$ be a stationary Markov chain defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with μ-initial distribution and $(\mathcal{X}, \mathcal{B})$ be the state space. A stochastic kernel P such that

$$Pf(X_k) = \mathbb{E}\left\{f(X_{k+1})/X_k\right\} \quad for \;\; k \geq 0 \tag{3.1}$$

with f be a bounded, measurable function on the state space.

In the sequel, we will denote

$$\begin{aligned} S_k f &= \sum_{i=0}^{k} P^i f, \\ S_n^* f &= \max_{0\leq k\leq n} S_k f \\ S^* f &= \sup_k S_k f. \end{aligned}$$

We will establish the ergodic theorem for operator P under measure μ. Firstly, we need the following theorem regarded as *Maximal Ergodic Theorem*

Theorem 3.1.1. ***(Maximal Ergodic Theorem)*** *For any $f \in L^1(\mu)$, we have*

$$\int\limits_{S^* f > 0} f \, d\mu \geq 0. \tag{3.2}$$

Proof. For $a \in \mathbb{R}$, set $a^+ = \max\{a, 0\}$. For any $a, b \in \mathbb{R}$ then

$$\max\{a, a+b\} = a + \max\{b, 0\} = a + b^+.$$

For two functions $g, h \in L^1(\mu)$

$$\begin{cases} P\left(\max\{g,h\}\right) = P\left(g + (h-g)^+\right) = P(g) + P\left((h-g)^+\right) \geq P(g) \\ \\ P\left(\max\{g,h\}\right) = P\left(h + (g-h)^+\right) = P(h) + P\left((g-h)^+\right) \geq P(h) \end{cases}$$

then

$$P\left(\max\{g,h\}\right) \geq \max\{P(g), P(h)\}.$$

And hence,

$$P(g^+) = P\left(\max\left[0, g\}\right) \geq \max\{P(0), P(g)\} = \max\{0, P(g)\} = \left[P(g)\right]^+.$$

Consider

$$\begin{aligned} S^*_{n+1}f &= \max_{0\leq k\leq n+1} S_k f = \max\left\{f, f+Pf, \ldots, f + \sum_{i=1}^{n+1} P^i f\right\} \\ &= f + \max\left\{0, \max_{0\leq k\leq n} P(S_k f)\right\} = f + \max\left\{P(0), \max_{k\leq n} P(S_k f)\right\} \\ &\leq f + P\left(\max\{0, \max_{k\leq n} S_k f\}\right) = f + P\left(\max\{0, S^*_n f\}\right) \\ &\leq f + P\left[(S^*_n f)^+\right]. \end{aligned}$$

Set $E_n = \{S^*_n f > 0\}$. Since

$$S^*_n f \leq f + P\left[(S^*_{n-1} f)^+\right] \leq f + P\left[(S^*_n f)^+\right]$$

then

$$\begin{aligned} \int\limits_{E_n} S^*_n f d\mu &\leq \int\limits_{E_n} f d\mu + \int\limits_{E_n} P\left[(S^*_n f)^+\right] d\mu \leq \int\limits_{E_n} f\, d\mu + \int\limits_{\mathbb{R}} P\left[(S^*_n f)^+\right] d\mu \\ &\leq \int\limits_{E_n} f d\mu + \int\limits_{\mathbb{R}} (S^*_n f)^+ d\mu = \int\limits_{E_n} f d\mu + \int\limits_{E_n} S^*_n f d\mu. \end{aligned}$$

It follows that,

$$\int\limits_{E_n} f d\mu \geq 0 \text{ for any } n \geq 0.$$

For $n \to \infty$, we obtain $\int\limits_{S^* f > 0} f\, d\mu \geq 0$ with $S^* f = \sup_k S_k f$. □

Corollary 3.1.1. *For any function $f \in L^1(\mu)$ then*

$$\int\limits_{M^* f > \alpha} f\, d\mu \geq \alpha \mu\{M^* f > \alpha\}, \tag{3.3}$$

with $M^* f = \sup_k |\frac{1}{k+1} S_k f|$.

Proof. By preceding theorem,

$$\int\limits_{M^*(f-\alpha)>0} (f-\alpha)\, d\mu = \int\limits_{M^*(f-\alpha)>0} f\, d\mu - \alpha\mu\{M^*(f-\alpha)>0\} \geq 0$$

and we get then

$$\int\limits_{M^* f>\alpha} f\, d\mu \geq \alpha\mu\{M^* f > \alpha\}.$$

□

Definition 3.1.1. *A Markov chain $(X_n)_{n\geq 1}$ is ergodic if $Ph = h$ for some $h \in L^1(\mu)$ then h is constant.*

Theorem 3.1.2. ***(Hopf's Ergodic Theorem)*** *If the chains $(X_n)_{n\geq 1}$ is ergodic then for any $g \in L^1(\mu)$*

$$\lim_{k\to\infty} \frac{1}{k+1} S_k g = \int g\, d\mu,\ \mu\ a.s.$$

Proof. Denote

$$M_k g = \frac{1}{k+1} S_k g \ \ and\ \ h = \lim_{k\to\infty} \inf_k M_k g$$

We decompose the proof into two steps:

Step 1. We consider the case $0 \leq g \leq 1$. Firstly, we show that h is a constant. For any $\ell \geq 0, P(\inf_{\ell\leq k} M_k g) \leq \inf_{\ell\leq k} M_k P g$ by Fatou's lemma. For $\ell \to \infty$, we have

$$\begin{aligned} Ph &\leq \lim_{\ell\to\infty} \inf_{\ell<k} (M_k P g) \leq \lim_{\ell\to\infty} \inf_{\ell<k} \left\{ M_k g + \frac{1}{k+1} P^{k+1} g - \frac{1}{k+1} g \right\} \\ &\leq \lim_{\ell\to\infty} \inf_{\ell<k} (M_k g) \leq h \end{aligned}$$

Since $0 \leq 1 - g \leq 1$, we have also: $P(1-h) \leq 1-h$ and so, $Ph \geq h$. Hence, $Ph = h$ and then h is constant by the ergodicity of the chains $(X_n)_{n\geq 1}$.

For any $\varepsilon > 0$, set

$$F = \left\{ h - \int g d\mu + \varepsilon < 0 \right\}$$

and

$$E = \left\{ \inf_k \left\{ S_k \left(g - \int g d\mu + \varepsilon \right) \right\} < 0 \right\}.$$

We will show that $F \subset E$ and then $\mu\{F\} = 0$. We have

$$\begin{aligned} h - \int g\, d\mu + \varepsilon &= \lim_{k\to\infty} \inf_k \frac{1}{k+1} S_k g - \int g\, d\mu + \varepsilon \\ &\geq \inf_k \frac{1}{k+1} S_k g - \int g\, d\mu + \varepsilon \\ &= \inf_k \frac{1}{k+1} S_k \left(g - \int g\, d\mu + \varepsilon \right) \end{aligned}$$

If $h - \int g d\mu + \varepsilon < 0$ then $\inf_k S_k \left(g - \int g\, d\mu + \varepsilon \right) < 0$. Hence, $F \subset E$. Moreover,

$$E = \left\{ \inf_k \left\{ S_k \left(g - \int g d\mu + \varepsilon \right) \right\} < 0 \right\} = \left\{ \sup_k \left\{ S_k \left(\int g d\mu - g - \varepsilon \right) \right\} > 0 \right\}.$$

By theorem 3.1.1, we have $\int\limits_E \left(\int g d\mu - \varepsilon - g \right) d\mu \geq 0$. It follows that

$$\int\limits_E \left(\int g d\mu - \varepsilon \right) d\mu \geq \int_E g d\mu \geq \int\limits_F g d\mu$$

and then $\left(\int g d\mu - \varepsilon \right) \mu \{E\} \geq \int\limits_F g d\mu$. Since h is a constant, $\mu \{F\}$ equals either 0 or 1. If $\mu \{F\} = 1$ then $\mu \{E\} = 1$ since $F \subset E$. Therefore $\int\limits_F g d\mu \leq \int g d\mu - \varepsilon, \ \forall \varepsilon > 0$. This is a contradiction! Hence, $\mu \{F\} = 0$. And for $\varepsilon \to 0$, $h \geq \int g d\mu$, μ a.s. We obtain

$$\lim_{k \to \infty} \inf_k \frac{1}{k+1} S_k g \geq \int g \, d\mu, \ \mu \ a.s. \tag{3.4}$$

Similarly, since $0 \leq 1 - g \leq 1$ we have

$$\frac{1}{k+1} S_k (1-g) = 1 - \frac{1}{k+1} S_k g$$

then

$$\frac{1}{k+1} S_k g = 1 - \frac{1}{k+1} S_k (1-g)$$

and so

$$\lim_{k \to \infty} \sup_k \frac{1}{k+1} S_k g = 1 - \lim_{k \to \infty} \inf_k \frac{1}{k+1} S_k (1-g) \leq 1 - \int (1-g) d\mu = \int g \, d\mu. \tag{3.5}$$

Combine (3.4) and (3.5), we obtain

$$\lim_{k \to \infty} \inf_k \frac{1}{k+1} S_k (g) \geq \int g \, d\mu \geq \lim_{k \to \infty} \sup_k \frac{1}{k+1} S_k g$$

Hence,

$$\lim_{k \to \infty} \frac{1}{k+1} S_k g = \int g \, d\mu.$$

Step 2. For any $g \in L^1(\mu)$, there exists $M > 0$ such that $|g| \leq M \ \ \mu \ a.s.$

Set $f = \frac{1}{2M}(g + M)$, then $0 \leq f \leq 1$. Applying Step 1, we have also

$$\lim_{k \to \infty} \frac{1}{k+1} S_k f = \int f \, d\mu, \ \mu \ a.s$$

and hence

$$\lim_{k \to \infty} \frac{1}{k+1} S_k g = \int g \, d\mu, \ \mu \ a.s$$

We finish the proof. □

Corollary 3.1.2. *For any $f \in L^1(\mu)$ such that $\int f \, d\mu = 0$ then*

$$\mu\{|Mf| > 0\} = 0,$$

with $Mf = \lim_{k\to\infty} M_k f$.

Proof. For any $\varepsilon > 0$ there exists $g \in L^\infty(\mu)$ such that $\|g - f\|_1 \leq \varepsilon$. We have

$$Mf = Mg + M(f-g)$$

then

$$|Mf| \leq |Mg| + M^*(f-g) \leq \left| \int g \, d\mu \right| + M^*(f-g) \leq \varepsilon + M^*(f-g).$$

For any $a > 0$, by corollary 3.1.1

$$\mu\{|Mf| > a\} \quad \leq \quad \mu\{M^*|f-g| > a - \varepsilon\} \leq \frac{\|f-g\|_1}{a-\varepsilon} \leq \frac{\varepsilon}{a-\varepsilon}.$$

For $\varepsilon \to 0$,

$$\mu\{|Mf| > a\} = 0.$$

For $a \to 0$,

$$\mu\{|Mf| > 0\} = 0.$$

□

3.2 Central limit theorem for stationary Markov chain

Suppose that $(X_n)_{n\geq 0}$ is stationary Markov chain with ν-initial stationary distribution and P is the transition probability of the chain. Define the operator Π on the space $L^\infty(\nu \otimes \nu)$ by

$$\Pi h(X_k) = \mathbb{E}\left\{h(X_k, X_{k+1})/X_k\right\}. \tag{3.6}$$

We consider again the theorem of Gordin-Lifshitz (1978).

Theorem 3.2.1. *For any $f \in L^2(\nu)$, set $g = Pf - f$ then we have*

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} g(X_k) \xrightarrow{D} \mathcal{N}\left(0, \sigma_g^2\right) \quad \text{as } n \longrightarrow \infty$$

where $\sigma_g^2 = \int f^2 d\nu - \int (Pf)^2 d\nu$.

Proof. Firstly, we decompose $g(X_k)$ as follows

$$g(X_k) = Pf(X_k) - f(X_{k+1}) + f(X_{k+1}) - f(X_k)$$

and then taking the sum on $k = 0, \dots, n-1$

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} g(X_k) \quad = \quad \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} [Pf(X_k) - f(X_{k+1})] + \frac{1}{\sqrt{n}} [f(X_n) - f(X_0)]$$

$$= \frac{1}{\sqrt{n}}\sum_{k=1}^{n} M_k + \frac{1}{\sqrt{n}}\left[f(X_n) - f(X_0)\right]$$

where $M_k = Pf(X_{k-1}) - f(X_k)$.

To prove this theorem, we have to prove that

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n} M_k \xrightarrow{D} \mathcal{N}\left(0, \sigma_g^2\right) \quad as\ n \longrightarrow \infty \tag{3.7}$$

and

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}}\left[f(X_n) - f(X_0)\right] = 0 \text{ in } L^2. \tag{3.8}$$

Set $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$, we see that M_n is $\mathcal{F}_n$-measurable and $\mathbb{E}\left\{M_n/\mathcal{F}_{n-1}\right\} = 0$ for any $n \geq 1$. We will show that the partial sums of M_n is a martingale with respect to $\mathcal{F}_n$ which satisfies the condition of Brown's theorem for martingale (theorem 2.3.2, chapter 2) and the remainder $\frac{1}{\sqrt{n}}\left[f(X_n) - f(X_0)\right]$ is negligible. This method is also called "martingale approximation" followed by several authors. For the most of this thesis, Brown's theorem mentions to theorem 2.3.2.

Since $f \in L^2(\nu)$, it is easy to see that (3.8) holds. So, it remains to prove (3.7). By setting

$$S_n = \sum_{k=1}^{n} M_k$$

for any $n \geq 1$, then S_n be a martingale with respect to $\mathcal{F}_n$ since

$$\mathbb{E}\left\{S_{n+1}/\mathcal{F}_n\right\} = S_n + \mathbb{E}\left\{M_{n+1}/\mathcal{F}_n\right\} = S_n$$

In order to prove (3.7), by Brown's theorem for martingale, we claim that

$$I_1 = \lim_{n\to\infty} \frac{1}{s_n^2}\sum_{k=1}^{n} \mathbb{E}\left\{M_k^2/\mathcal{F}_{k-1}\right\} = 1, \tag{3.9}$$

and the second one

$$I_2 = \lim_{n\to\infty} \frac{1}{s_n^2}\sum_{k=1}^{n} \mathbb{E}\left\{M_k^2 \mathbf{1}_{\{|M_k|>\delta s_n\}}/\mathcal{F}_{k-1}\right\} = 0, \quad \forall\delta > 0 \tag{3.10}$$

where $s_n^2 = \sum\limits_{k=1}^{n} \mathbb{E}\left\{M_k^2\right\}$.

Proof of (3.9). For each $k \geq 1$, one has

$$\mathbb{E}\left\{M_k^2/\mathcal{F}_{k-1}\right\} = Pf^2(X_{k-1}) - (Pf)^2(X_{k-1}) = \psi(X_{k-1}).$$

where $\psi = Pf^2 - (Pf)^2$, and

$$s_n^2 = \sum_{k=1}^{n} \mathbb{E}\left\{M_k^2\right\} = \mathbb{E}\left\{\sum_{k=1}^{n} \mathbb{E}\left\{M_k^2/\mathcal{F}_{k-1}\right\}\right\} = \sum_{k=0}^{n-1} \mathbb{E}\left\{Pf^2(X_k) - (Pf)^2(X_k)\right\}$$

$$= \sum_{j=0}^{n-1} \int \left[Pf^2 - (Pf)^2\right] d\nu = n \int \psi d\nu.$$

The law of large number for stationary $(Y_n = \psi(X_n))_{n\geq 0}$ ensures that

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\left\{M_k^2/\mathcal{F}_{k-1}\right\} &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi(X_k) = \mathbb{E}\left\{\psi(X_0)\right\} \\ &= \int \psi d\nu, \text{ in } L^1.\end{aligned}$$

We have thus proved

$$I_1 = \lim_{n\to\infty} \frac{\frac{1}{n}\sum\limits_{k=0}^{n-1} \psi(X_k)}{\frac{1}{n} s_n^2} = \frac{\int \psi \, d\nu}{\int \psi \, d\nu} = 1.$$

and we finished the proof of (3.9).

Proof of (3.10). Fix $M > 0$, put

$$\psi_M = \Pi h_M,$$

where $h_M(x,y) = [Pf(x) - f(y)]^2 \, \mathbb{1}_{\{|Pf(x)-f(y)|>\delta M\}}$. One has

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\left\{M_k^2 \mathbb{1}_{\{|M_k|>\delta M\}}/\mathcal{F}_{k-1}\right\} &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi_M(X_k) = \mathbb{E}\left\{\psi_M(X_0)\right\} \\ &= \int \psi_M d\nu.\end{aligned}$$

Since $\lim\limits_{n\to\infty} s_n = +\infty$, there exists $N > 0$ such that for any $n > N$ then $s_n > M$. And therefore,

$$\frac{1}{s_n^2} \sum_{k=N+1}^{n} \mathbb{E}\left\{M_k^2 \mathbb{1}_{\{|M_k|>\delta s_n\}}/\mathcal{F}_{k-1}\right\} \leq \frac{1}{s_n^2} \sum_{k=N+1}^{n} \mathbb{E}\left\{M_k^2 \mathbb{1}_{\{|M_k|>\delta M\}}/\mathcal{F}_{k-1}\right\}$$

for any $n > N$. For $n \to \infty$, we obtain

$$I_2 \leq \frac{\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} \mathbb{E}\left\{M_k^2 \mathbb{1}_{\{|M_k|>\delta M\}}/\mathcal{F}_{k-1}\right\}}{\lim\limits_{n\to\infty} \frac{s_n^2}{n}} = \frac{\int \psi_M d\nu}{\int \psi d\nu}$$

Since $\lim\limits_{M\to\infty} \int \psi_M d\nu = 0$, then

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{n} \mathbb{E}\left\{M_k^2 \mathbb{1}_{\{|M_k|>\delta s_n\}}/\mathcal{F}_{k-1}\right\} = 0 \text{ in } L^1$$

and we finished the proof of (3.10).

Finally, by Brown's theorem (theorem 2.3.2) for martingale

$$\frac{1}{s_n}\sum_{k=1}^{n} M_k \xrightarrow{D} \mathcal{N}(0,1)$$

it follows that

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n} M_k \xrightarrow{D} \mathcal{N}\left(0,\sigma_g^2\right)$$

since $s_n = \sigma_g\sqrt{n}$. □

3.3 Rewrite the preceding proof for the framework of shift

We recall here $(X_n)_{n\geq 0}$ be a stationary Markov chain defined on a probability space $(\Omega,\mathcal{A},\mathbb{P})$ with ν-initial distribution, P is a transition probability and $(\mathcal{X},\mathcal{B})$ be the state space.

We construct a preserving-measure system $(\mathbb{R}^{\mathbb{N}},\mathbb{B},\mathbb{P}_\nu,\sigma)$ by

$$\begin{array}{rccc} \sigma: & \mathbb{R}^{\mathbb{N}} & \longrightarrow & \mathbb{R}^{\mathbb{N}} \\ & x & \longmapsto & \sigma x, \end{array}$$

such that $(\sigma x)_n = x_{n+1}$.

Define

$$\mathbb{P}_\nu\{x_0\in A_0,\ldots,x_r\in A_r\} = \int\limits_{A_0}\nu(dx_0)\int\limits_{A_1}P(x_0,dx_1)\ldots\int\limits_{A_{r-1}}P(x_{r-2},dx_{r-1})P(x_{r-1},A_r)$$

Define π_n be the projection onto the nth coordinate of $\mathbb{R}^{\mathbb{N}}$

$$\begin{array}{rccc} \pi_n: & \mathbb{R}^{\mathbb{N}} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \pi_n x = x_n. \end{array}$$

Since $(\pi_n)_n$ has the same joint distribution on $\mathbb{R}^{\mathbb{N}}$ as $(X_n)_n$ on Ω, then $(\pi_0\circ\sigma^n)_n$ has the same joint distribution on $\mathbb{R}^{\mathbb{N}}$ as $(X_n)_n$ on Ω. In the sequel of this section and the next one, we will assume that $\Omega=\mathbb{R}^{\mathbb{N}}$ and $X_k=\pi_0\circ\sigma^k$.

For any $f\in L^2(\nu)$, let $g=Pf-f$. We have

$$\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} g(X_k) = \frac{1}{\sqrt{n}}\sum_{k=1}^{n} M_k + \frac{1}{\sqrt{n}}\left[f(X_n)-f(X_0)\right] \tag{3.11}$$

where $M_k = Pf(X_{k-1}) - f(X_k)$.

Now, we want to show that $(M_k)_{k\geq 1}$ satisfies the condition of Brown's theorem. Set $\mathcal{F}_n=\sigma(X_0,X_1,\ldots,X_n)$, we see that M_n is $\mathcal{F}_n$-measurable and $\mathbb{E}\left\{M_n/\mathcal{F}_{n-1}\right\}=0$ for any $n\geq 1$. It remains to check coditions (3.9) and (3.10) in the preceding section.

Proof of (3.9). Set $s_n^2 = \sum_{k=1}^{n} \mathbb{E}\left\{M_k^2\right\}$. We have

$$\begin{aligned} s_n^2 &= \mathbb{E}\left\{\sum_{k=1}^{n} \mathbb{E}\left\{M_k^2/\mathcal{F}_k\right\}\right\} = \sum_{k=0}^{n-1} \mathbb{E}\left\{Pf^2(X_k) - (Pf)^2(X_k)\right\} \\ &= \sum_{j=0}^{n-1} \int \left[Pf^2 - (Pf)^2\right] d\nu = n \int \phi \, d\mathbb{P}_\nu, \end{aligned} \tag{3.12}$$

where $\phi = \left[P(f^2) - (Pf)^2\right] \pi_0$.

For each $k = 1, 2, \ldots$

$$\begin{aligned} \mathbb{E}\left\{M_k^2/\mathcal{F}_{k-1}\right\} &= P(f^2)(X_{k-1}) - (Pf)^2(X_{k-1}) \\ &\approx Pf^2(\pi_0 \circ \sigma^{k-1}) - (Pf)^2(\pi_0 \circ \sigma^{k-1}) \\ &= \left[P(f^2) - (Pf)^2\right] \pi_0 \circ \sigma^{k-1}. \end{aligned}$$

Taking the sum on $k = 1, 2, \ldots, n$

$$\sum_{k=1}^{n} \mathbb{E}\left\{M_k^2/\mathcal{F}_{k-1}\right\} = \sum_{k=0}^{n-1} \phi \circ \sigma^k.$$

For $n \to \infty$, to treat this limit, we use the ergodic theorem with σ be measure preserving transformation. To do that, we must show that $\phi \in L^1(\nu)$, i.e $\int |\phi| d\nu < \infty$. One has

$$\begin{aligned} \int |\phi| \, d\mathbb{P}_\nu &= \int \left|\left[Pf^2 - (Pf)^2\right] \pi_0\right| \, d\mathbb{P}_\nu = \int \left|\left[Pf^2 - (Pf)^2\right]\right| d\nu \\ &\leq \int \left|Pf^2\right| d\nu + \int \left|(Pf)^2\right| d\nu \leq \int f^2 d\nu + \int Pf^2 d\nu < \infty. \end{aligned}$$

By ergodic theorem

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ \sigma^k = \int \phi \, d\mathbb{P}_\nu. \tag{3.13}$$

Combine (3.12) and (3.13) then $I_1 = 1$ which completes the proof of (3.9).

Proof of (3.10). Fix $M > 0$, put

$$\phi_M(x) = \Pi h_M(x),$$

with $h_M(x, y) = \left[Pf(x) - f(y)\right]^2 \mathbb{1}_{\{|Pf(x) - f(y)| > \delta M\}}$. One has

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\left\{M_k^2 \mathbb{1}_{\{|M_k| > \delta M\}}/\mathcal{F}_{k-1}\right\} &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi_M(X_k) = \mathbb{E}\left\{\phi_M(X_0)\right\} \\ &= \int \phi_M d\nu, \quad \nu \; a.s. \end{aligned}$$

Since $\lim_{n\to\infty} s_n = +\infty$, there exists $N > 0$ such that for any $n > N$ then $s_n > M$. And therefore,

$$\frac{1}{s_n^2} \sum_{k=N+1}^{n} \mathbb{E}\left\{M_k^2 \mathbb{1}_{\{|M_k| > \delta s_n\}}/\mathcal{F}_{k-1}\right\} \leq \frac{1}{s_n^2} \sum_{k=N+1}^{n} \mathbb{E}\left\{M_k^2 \mathbb{1}_{\{|M_k| > \delta M\}}/\mathcal{F}_{k-1}\right\}$$

for any $n > N$. For $n \to \infty$, we obtain

$$I_2 \quad \leq \quad \frac{\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=1}^{n} \mathbb{E}\left\{M_k^2 \mathbb{1}_{\{|M_k|>\delta M\}}/\mathcal{F}_{k-1}\right\}}{\lim\limits_{n\to\infty} \frac{s_n^2}{n}} = \frac{\int \phi_M d\nu}{\int \phi \, d\mathbb{P}_\nu}.$$

Since $\lim\limits_{M\to\infty} \int \phi_M d\nu = 0$, then $I_2 = 0$, which completes the proof of (3.10).

3.4 Central limit theorem for Markov chain started at a point

Suppose $(X_n)_{n\geq 0}$ be a sequence of ergodic stationary Markov chain. In section 3.2, we supposed that there is a transition probability P such that

$$Pf(X_k) = \mathbb{E}\left\{f(X_{k+1})/X_k\right\}$$

and there exist a probability measure ν is $P-$invariant. Now, we consider here the case $X_0 = x_0$ fixed. Let $f \in L^2(\nu)$, set $g = f - Pf$. Using martingale approximation method, we claim that $S_n = \sum\limits_{k=0}^{n-1} g(X_k)$ be also asymptotic normality.

We construct a preserving-measure system $(\mathbb{R}^{\mathbb{Z}}, \mathbb{B}, \mathbb{P}_{x_0}, \sigma)$

$$\begin{array}{rccc} \sigma: & \mathbb{R}^{\mathbb{Z}} & \longrightarrow & \mathbb{R}^{\mathbb{Z}} \\ & x & \longmapsto & \sigma x, \end{array}$$

such that $(\sigma x)_n = x_{n+1}$. Define

$$\mathbb{P}_{x_0}\{x_0 \in A_0, \dots, x_r \in A_r\} = \delta_{x_0}(A_0) \int\limits_{A_1} P(x_0, dx_1) \dots \int\limits_{A_{r-1}} P(x_{r-2}, dx_{r-1}) P(x_{r-1}, A_r)$$

where δ_{x_0} be the unit mass concentrated at x_0

$$\delta_{x_0}(A_0) = \begin{cases} 1 & if \quad x_0 \in A_0 \\ 0 & if \quad x_0 \notin A_0. \end{cases}$$

Define π_n be the projection onto the nth coordinate of $\mathbb{R}^{\mathbb{Z}}$

$$\begin{array}{rccc} \pi_n: & \mathbb{R}^{\mathbb{Z}} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & \pi_n x = x_n. \end{array}$$

Since $(\pi_n)_n$ has the same joint distribution on $\mathbb{R}^{\mathbb{Z}}$ as $(X_n)_n$ on Ω, so $(\pi_0 \circ \sigma^n)_n$ has the same joint distribution on $\mathbb{R}^{\mathbb{Z}}$ as $(X_n)_n$ on Ω.

Theorem 3.4.1. *For any $f \in L^2(\nu)$, set $g = Pf - f$ then we have*

$$\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} g(X_k) \xrightarrow{\quad D \quad} \mathcal{N}\left(0, \sigma_g^2\right) \quad as\ n \longrightarrow \infty$$

where $\sigma_g^2 = \int f^2 d\nu - \int (Pf)^2 d\nu$.

Proof. Decomposing $g(X_k)$ as the preceding section, we obtain also

$$\begin{aligned} \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1} g(X_k) &= \frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}[Pf(X_k)-f(X_{k+1})]+\frac{1}{\sqrt{n}}[f(X_n)-f(X_0)] \\ &= \frac{1}{\sqrt{n}}\sum_{k=1}^{n} M_k+\frac{1}{\sqrt{n}}[f(X_n)-f(X_0)] \end{aligned}$$

by putting $M_k = Pf(X_{k-1}) - f(X_k)$.

Proposition 3.4.1. *We have*

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}[f(X_n)-f(X_0)] = 0, \ \ \mathbb{P}_\nu \ a.a. \tag{3.14}$$

Proof. To prove this proposition, we need the following lemma:

Lemma 3.4.1. *For any $g \in L^1(\mu)$ and $g \geq 0$, then $\sum_{n=1}^{\infty}\mu\{g > n\} \leq \int g \, d\mu$.*

Since this lemma is basic, we skip the proof here to concentrate on the proposition. By lemma 3.4.1, then for any $\varepsilon > 0$

$$\sum_{n=1}^{\infty}\mathbb{P}_\nu\left\{\frac{1}{\sqrt{n}}f\circ\sigma^n > \varepsilon\right\} = \sum_{n=1}^{\infty}\mathbb{P}_\nu\left\{\frac{f^2}{\varepsilon^2} > n\right\} \leq \frac{1}{\varepsilon^2}\int f^2 \, d\mathbb{P}_\nu$$

By Borel Cantelli 's lemma, we obtain

$$\mathbb{P}_\nu\left\{\overline{\lim_{n\to\infty}}\frac{1}{\sqrt{n}}f\circ\sigma^n > \varepsilon\right\} = 0,$$

and hence,

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}f\circ\sigma^n = 0, \ \ \mathbb{P}_\nu \ a.a$$

This implies

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}f(X_{n+1}) = 0, \ \ \mathbb{P}_\nu \ a.a.$$

□

Proposition 3.4.2. *For $n \to \infty$, the following asymptotic normality holds*

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n} M_k \xrightarrow{D} \mathcal{N}\left(0, \sigma_g^2\right) \tag{3.15}$$

Proof. Set $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$, we see that M_n is $\mathcal{F}_n$-measurable. We will show that the partial sums of M_n is a martingale with respect to $\mathcal{F}_n$ which satisfies the condition of Brown 's Theorem.

It is easy to check that $\mathbb{E}_{x_0}\{M_n/\mathcal{F}_{n-1}\} = 0$ for any $n \geq 1$ and hence the partial sums of M_n is a martingale with respect to $\mathcal{F}_n$. The next step, we will treat the following statements:

$$I_1 = \lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{n} \mathbb{E}_{x_0} \left\{ M_k^2 / \mathcal{F}_{k-1} \right\} = 1, \tag{3.16}$$

and

$$I_2 = \lim_{n\to\infty} \frac{1}{s_n^2} \sum_{k=1}^{n} \mathbb{E}_{x_0} \left\{ M_k^2 \mathbb{1}_{\{|M_k| > \delta s_n\}} / \mathcal{F}_{k-1} \right\} = 0, \quad \forall \delta > 0 \tag{3.17}$$

where $s_n^2 = \sum\limits_{k=1}^{n} \mathbb{E}_{x_0}\left\{M_k^2\right\}$. Since $(X_n)_{n\geq 0}$ is a Markov chain, the conditional expectation in (3.16) and (3.17) does not depend on x_0. It will be denoted by $\mathbb{E}\left\{\bullet/\mathcal{F}_{k-1}\right\}$ in the sequel.

Proof of (3.16). Let begin with the calculator of s_n^2

$$\begin{aligned}
s_n^2 &= \sum_{k=1}^{n} \mathbb{E}_{x_0}\left\{M_k^2\right\} = \sum_{k=1}^{n} \mathbb{E}_{x_0}\left\{\mathbb{E}(M_k^2/\mathcal{F}_{k-1})\right\} \\
&= \sum_{k=0}^{n-1} \mathbb{E}_{x_0}\left\{Pf^2(X_k) - (Pf)^2(X_k)\right\} = \sum_{k=0}^{n-1} \mathbb{E}_{x_0}\left\{\left[Pf^2 - (Pf)^2\right]\pi_0 \circ \sigma^k\right\} \\
&= \sum_{k=0}^{n-1} \mathbb{E}_{x_0}\left\{\phi \circ \sigma^k\right\} = \sum_{k=0}^{n-1} P^k \psi(x_0)
\end{aligned}$$

where $\phi = \left[Pf^2 - (Pf)^2\right]\pi_0 = \psi\pi_0$.

For each $k = 1, 2, \ldots, n$

$$\begin{aligned}
\mathbb{E}\left\{M_k^2/\mathcal{F}_{k-1}\right\} &= Pf^2(X_{k-1}) - (Pf)^2(X_{k-1}) \\
&\approx Pf^2(\pi_0 \circ \sigma^{k-1}) - (Pf)^2(\pi_0 \circ \sigma^{k-1}) \\
&= \phi \circ \sigma^{k-1}.
\end{aligned}$$

It follows that

$$I_1 = \frac{\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n-1} \phi \circ \sigma^k}{\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n-1} P^k\psi \circ \pi_0} = \frac{\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=0}^{n-1} \phi \circ \sigma^k}{\lim\limits_{n\to\infty} \frac{1}{n} \sum\limits_{k=0}^{n-1} P^k\psi \circ \pi_0} = \frac{\int \phi \, d\mathbb{P}_\nu}{\int \phi \, d\mathbb{P}_\nu} = 1, \ \mathbb{P}_\nu \ a.a\ x$$

which completes the proof of (3.16).

Proof of (3.17). Fix $M > 0$, put

$$\phi_M(x) = \Pi h_M(x),$$

where the function $_M(x, y)$ is defined by

$$h_M(x, y) = \left[Pf(x) - f(y)\right]^2 \mathbb{1}_{\{|Pf(x) - f(y)| > \delta M\}}.$$

One has

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\left\{M_k^2 \mathbb{1}_{\{|M_{k+1}| > \delta M\}} / \mathcal{F}_{k-1}\right\} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi_M(X_k) = \mathbb{E}\left\{\phi_M(X_0)\right\}$$

$$= \int \phi_M d\nu, \quad \nu \ a.s$$

Since $\lim\limits_{n\to\infty} s_n = +\infty$, there exists $N > 0$ such that $\forall n > N$ then $s_n > M$. And therefore,

$$\frac{1}{s_n^2}\sum_{k=N+1}^{n} \mathbb{E}\left\{M_k^2 \mathbb{1}_{\{|M_{k+1}|>\delta s_n\}}/\mathcal{F}_{k-1}\right\} \leq \frac{1}{s_n^2}\sum_{k=N+1}^{n} \mathbb{E}\left\{M_k^2 \mathbb{1}_{\{|M_k|>\delta M\}}/\mathcal{F}_{k-1}\right\}$$

for any $n > N$. For $n \to \infty$, we obtain

$$I_2 \quad \leq \quad \frac{\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{k=1}^{n} \mathbb{E}\left\{M_k^2 \mathbb{1}_{\{|M_k|>\delta M\}}/\mathcal{F}_{k-1}\right\}}{\lim\limits_{n\to\infty} \frac{s_n^2}{n}} = \frac{\int \phi_M d\nu}{\int \phi \ d\mathbb{P}_\nu}$$

Since $\lim\limits_{M\to\infty} \int \phi_M d\nu = 0$, then $I_2 = 0$, which completes the proof of (3.9). □

Finally, by Brown's theorem for martingale

$$\frac{1}{s_n}\sum_{k=1}^{n} M_k \xrightarrow{D} \mathcal{N}(0,1)$$

it follows that

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n} M_k \xrightarrow{D} \mathcal{N}\left(0,\sigma_g^2\right)$$

since $s_n = \sigma_g\sqrt{n}$. □

Chapter 4

Central limit theorem for Random walk in Random environment based on martingale approximation

4.1 Introduction

4.1.1 Random environment and random walks

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. The definition of a Random walk in Random environment involves two ingredients:

- The environment which is randomly chosen but remains fixed throughout the time evolution.
- The random walk whose transition probability are determined by the environment.

The space Ω is interpreted as the space of environments. For each $\omega \in \Omega$, we define the random walk in the environment ω as the (time-homogeneous) Markov chain $\{X_n, n = 0, 1, 2, \ldots\}$ on $\mathbb{Z}^d$ with certain (random) transition probabilites

$$p(x, y, \omega) = \mathbb{P}_\omega\{X_1 = y / X_0 = x\}. \tag{4.1}$$

The probability measure $\mathbb{P}_\omega$ that determines the distribution of the random walk in a given environment ω. In the case the random walk with the initial condition $X_0 = x$,

$$\mathbb{P}^x_\omega\{X_0 = x\} = 1. \tag{4.2}$$

The probability measure $\mathbb{P}^x_\omega$ indicates the distribution of the random walk in a given environment ω with the initial position of the walk is referred to as *the Quenched law.*

By averaging the Quenched probability $\mathbb{P}^x_\omega$ further, with respect to the environment distribution, we obtain *the Annealed measure* $\mathbf{P}^x = \mathbb{P} \times \mathbb{P}^x_\omega$, which determines the probability law of the random walk in random environment

$$\mathbf{P}^x(A) = \int_\Omega \mathbb{P}^x_\omega(A)\mathbb{P}(d\omega) = \mathbb{E}\left\{\mathbb{P}^x_\omega(A)\right\}. \tag{4.3}$$

Expectation with respect to the Annealed measure $\mathbf{P}^x$ will be denoted by $\mathbf{E}^x$.

***Remark* 4.1.1.** *If some property A of the random walk in random environment holds almost surely with respect to the Quenched law $\mathbb{P}_\omega^x$ for almost all environments, then this property is also true with probability one under the Annealed law $\boldsymbol{P}^x$.*

In the sequel of this chapter, it is devoted to the Quenched version. We will establish the Quenched CLT for reversible random walk in random environment in one dimension. Our proof is to use martingale approximation for the random walk.

4.1.2 Presentation of the model-dimension one

4.1.2.1 Site randomnes

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. One chooses i.i.d. variables $p(x,\omega)$, $x \in \mathbb{Z}$, with value in $[0,1]$, $q(x,\omega) = 1 - p(x,\omega)$, and for a given realization ω of the environment , one considers a Markov chain X_n on $\mathbb{Z}$, which has probability $p(x,\omega)$ of jumping to the right neighbor $x+1$ and $q(x,\omega)$ of jumping to the left neighbor $x-1$, given it is located in x. This is the so-called *random walk in random environment in one dimension.*

4.1.2.2 Bond randomness

One now chooses i.i.d. variables $c_{x,x+1}(\omega)$, $x \in \mathbb{Z}$, with value in $(0,+\infty)$, and for a given realization ω of the environment , X_n is a Markov chain on $\mathbb{Z}$, performing jumps to nearest neighbors with a transition kernel determined by

$$p(x,\omega) = \frac{c_{x,x+1}(\omega)}{c_{x-1,x}(\omega) + c_{x,x+1}(\omega)}. \tag{4.4}$$

The quantity $c_{x,x+1}(\omega)$ is the so-called conductance of the edge between $\{x, x+1\}$ in the environment "ω".

4.1.3 The environment viewed from the particle

The basic idea is to focus on the evolution of the environment viewed from the current position of the walk. More specifically in the case of bond randomness, for $0 < a < b < \infty$,

- $\Omega = [a,b]^C$ with $C = \{\{x,x+1\}, x \in \mathbb{Z}\}$, the set of nearest neighbor bonds on $\mathbb{Z}$, endowed with the canonical product σ-field $\mathcal{B}$.
- $\mathbb{P}$: a product measure on Ω, making the canonical coordinates i.i.d.
- T^x, $x \in \mathbb{Z}$, the canonical translations on Ω :

$$(T^y\omega)(\{x,x+1\}) = \omega(\{x+y, x+y+1\}). \tag{4.5}$$

- $\mathbb{P}_\omega^x$, $x \in \mathbb{Z}$, the canonical law of the Markov chain on $\mathbb{Z}$ with transition probability described by (4.4) with $c_{x,x+1}(\omega) = \omega(\{x,x+1\})$.

The environment viewed from the particle is the ω-value process

$$\overline{\omega}_n = T^{X_n}\omega, \quad n \geq 0. \tag{4.6}$$

Under $\mathbb{P}^0_\omega$, $\omega \in \Omega$, $\overline{\omega}_n$ is a Markov chain with state space Ω and transition kernel:

$$Pf(\omega) = p(0,\omega)f(T\omega) + q(0,\omega)f(T^{-1}\omega) \tag{4.7}$$

with f bounded measurable on Ω.

4.2 CLT for Reversible Random Walks in Random environment

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space and T is an invertible measure preserving transformation on Ω which is ergodic. More precisely, T acts on Ω by

$$\begin{array}{rccc} T: & \Omega \times \mathbb{Z} & \longrightarrow & \Omega \\ & (\omega, k) & \longmapsto & T^k\omega, \end{array}$$

which is joint measurable and satisfies

- For any $k, h \in \mathbb{Z}: T^{k+h} = T^kT^h$ and $T^0\omega = \omega$.
- T preserves the measure $\mu : \mu(T^kA) = \mu(A)$ for any $k \in \mathbb{Z}$.
- T is ergodic: If $T^kA = A$ (up to null sets) for some $k \in \mathbb{Z}$ then $\mu(A) = 0$ or 1.

For $k \in \mathbb{Z}$, we define a conductivity of the edge between $\{k, k+1\}$ is $c(T^k\omega)$ and $\{k, k-1\}$ is $c(T^{k-1}\omega)$, which c be a positive measurable function on Ω. We refer to ω as an environment since each ω in Ω determines a conductivity for all edges of $\mathbb{Z}$. The space Ω is interpreted as the space of environments.

Fix $\omega \in \Omega$, we consider a random walk $(X_n)_{n\geq 0}$ on $\mathbb{Z}$ which $X_0 = 0$ and its transition probability $p(\omega, k, h)$ given by

$$p(\omega; k, k+1) = \frac{c(T^k\omega)}{\bar{c}(T^k\omega)} \text{ and } p(\omega; k, k-1) = \frac{c(T^{k-1}\omega)}{\bar{c}(T^k\omega)}, \tag{4.8}$$

where $\bar{c}(\omega) = c(\omega) + c(T^{-1}\omega)$. The set of possible jumps will be denoted by $\Lambda = \{-1, 1\}$ and for $y \in \Lambda$ we abbreviate $p(\omega; 0, y) = p(\omega; y)$. These random walks are reversible since $\bar{c}(T^x\omega)p(\omega; x, y) = \bar{c}(T^y\omega)p(\omega; y, x)$ for all adjacent vertices x, y in $\mathbb{Z}$.

We note that random walk X_n depend on the property of function c. In the sequel of this chapter, we will establish the Quenched CLT for $(X_n)_{n\geq 0}$. The method is to use martingale approximation. It is also adapted from Kozlov ([31], 1985) and Daniel Boivin ([7], 1993).

Theorem 4.2.1. *For almost all environment ω,*

$$\frac{X_n}{\sqrt{n}} \xrightarrow{\ D\ } \mathcal{N}\left(0, \sigma^2\right) \qquad \textit{as } n \longrightarrow \infty$$

if c and $c^{-1} \in L^1(\mu)$, where $\sigma^2 = \left[\int \frac{1}{c}\, d\mu \int c\, d\mu\right]^{-1}$.

To prove this theorem, we define a real additive 1-cocycle of the action T be a real measurable function

$$\begin{array}{rccc} F: & \Omega \times \mathbb{Z} & \longrightarrow & \mathbb{R} \\ & (\omega, k) & \longmapsto & F(\omega, k), \end{array}$$

such that

$$F(\omega, k+h) = F(\omega, k) + F(T^k\omega, h)$$

and

$$\begin{cases} F(\omega, 1) &= \frac{1}{c(\omega)} \\ F(\omega, 0) &= 0. \end{cases}$$

By the definition of F, one has

$$F(\omega, k) = \begin{cases} \sum\limits_{i=0}^{k-1} \frac{1}{c(T^i\omega)} & if \quad k \geq 1 \\ 0 & if \quad k = 0 \\ -\sum\limits_{i=1}^{-k} \frac{1}{c(T^{-i}\omega)} & if \quad k \leq -1. \end{cases}$$

and by the pointwise ergodic theorem

$$\lim_{m \to \infty} \frac{F(\omega, m)}{m} = \int \frac{1}{c} d\mu, \quad \mu \ a.a \ \omega. \tag{4.9}$$

It follows that $\frac{1}{\int \frac{1}{c} d\mu} F(\omega, m) \sim m$. Therefore, we will decompose X_n as follows

$$\frac{X_n}{\sqrt{n}} = \frac{1}{\int \frac{1}{c} d\mu} \frac{F(\omega, X_n)}{\sqrt{n}} + \frac{1}{\sqrt{n}} \left(X_n - \frac{1}{\int \frac{1}{c} d\mu} F(\omega, X_n) \right). \tag{4.10}$$

Set $M_n = F(\omega, X_n)$. Fix $\omega \in \Omega$ and let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$, we point out $(M_n)_{n \geq 0}$ is a martingale with respect to $\mathcal{F}_n$ and $X_n - \frac{1}{\int \frac{1}{c} d\mu} F(\omega, X_n)$ defines a cocycle of nul expectation. Furthermore, we claim that $\frac{M_n}{\sqrt{n}}$ be asymptotic normality.

Proposition 4.2.1. *For almost environment ω,*

$$\frac{M_n}{\sqrt{n}} \xrightarrow{D} \mathcal{N}\left(0, \frac{\int \frac{1}{c} d\mu}{\int c d\mu}\right) \qquad as\ n \longrightarrow +\infty.$$

Proof. We shall show that $(M_n)_{n \geq 0}$ sastifies the conditions of Brown's theorem for martingale (theorem 2.3.2).

Fix $\omega \in \Omega$, let $Y_n = M_n - M_{n-1}$ for any $n \geq 1$, then $M_n = \sum_{i=1}^n Y_i$ since $M_0 = F(\omega, 0) = 0$. One has

$$\begin{aligned} \mathbb{E}_\omega \{Y_n / \mathcal{F}_{n-1}\} &= \mathbb{E}_\omega \{(M_n - M_{n-1}) / X_{n-1} = k\} \\ &= \frac{1}{c(T^k\omega)} \frac{c}{\bar{c}}(T^k\omega) - \frac{1}{c(T^{k-1}\omega)} \frac{c(T^{k-1}\omega)}{\bar{c}(T^k\omega)} = 0 \end{aligned}$$

then $(M_n)_{n\geq 0}$ is a martingale with respect to $\mathcal{F}_n$. Let $s_n^2 = \sum_{i=1}^{n} \mathbb{E}_\omega \left\{Y_i^2\right\}$. By Brown's theorem, the proposition 4.2.1 will be proved if the following conditions hold:

$$I_1 = \lim_{n\to\infty} \frac{1}{s_n^2} \sum_{i=1}^{n} \mathbb{E}_\omega \left\{Y_i^2/\mathcal{F}_{i-1}\right\} = 1, \tag{4.11}$$

and

$$I_2 = \lim_{n\to\infty} \frac{1}{s_n^2} \sum_{i=1}^{n} \mathbb{E}_\omega \left\{Y_i^2 \mathbb{1}_{\{|Y_i|>\delta s_n\}}/\mathcal{F}_{i-1}\right\} = 0 \tag{4.12}$$

for any $\delta > 0$.

Proof of (4.11). We introduce the *left shift* $\sigma : \Omega^{\mathbb{N}} \to \Omega^{\mathbb{N}}$ such that $\forall \tilde{\omega} = (\omega_i) \in \Omega^{\mathbb{N}}$ then $(\sigma\tilde{\omega})_i = \omega_{i+1}$. The shift σ is a measure preserving on $\Omega^{\mathbb{N}}$.

Let us build for any probability measure ν on Ω a probability measure $\mathbb{P}_\nu$ on $\Omega^{\mathbb{N}}$ by

$$\mathbb{P}_\nu(\tilde{\omega}) = \nu(\omega_0) \otimes \mathbb{P}_{\omega_0}(\omega_1, \omega_2, \ldots).$$

The projection onto the nth coordinate of $\Omega^{\mathbb{N}}$ is defined by

$$\begin{array}{rccc} \pi_n : & \Omega^{\mathbb{N}} & \longrightarrow & \Omega \\ & \tilde{\omega} & \longmapsto & \pi_n\tilde{\omega} = \omega_n. \end{array}$$

One has

$$\begin{aligned} \mathbb{E}_\omega \left\{Y_n^2/X_{n-1} = k\right\} &= \frac{1}{c^2(T^k\omega)} \cdot \frac{c}{\bar{c}}(T^k\omega) + \frac{1}{c^2(T^{k-1}\omega)} \cdot \frac{c(T^{k-1}\omega)}{\bar{c}(T^k\omega)} \\ &= \left(\frac{1}{c(T^k\omega)} + \frac{1}{c(T^{k-1}\omega)}\right) \frac{1}{\bar{c}}(T^k\omega) \\ &= \varphi(T^k\omega) \end{aligned}$$

where $\varphi = \left(\frac{1}{c} + \frac{1}{c(T^{-1})}\right)\frac{1}{\bar{c}}$. Hence,

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_\omega \left\{Y_i^2/\mathcal{F}_{i-1}\right\} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_\omega \left\{Y_i^2/X_{i-1}\right\} = \frac{1}{n}\sum_{i=1}^{n} \varphi\left(T^{X_{i-1}}\omega\right). \tag{4.13}$$

We want to use Birkhoff's ergodic theorem to treat the limit of the right hand side in (4.13). To to this, we have to show that $\left(T^{X_n}\omega\right)_{n\geq 0}$ with initial law $d\nu(\omega) = \frac{\bar{c}(\omega)}{\int \bar{c} d\mu} d\mu(\omega)$ be a stationary ergodic Markov chain.

Consider the process of the environment viewed from the particle $(W_n)_{n\geq 0}$ on Ω defined by

$$W_n = T^{X_n}\omega \text{ and } W_0 = \omega \tag{4.14}$$

then it is a Markov chain with the transition probabilities

$$\left\{\begin{aligned} \mathbb{P}\left\{(W_n = T\omega)/(W_{n-1} = \omega)\right\} &= \frac{c(\omega)}{\bar{c}(\omega)}, \\ \mathbb{P}\left\{(W_n = T^{-1}\omega)/(W_{n-1} = \omega)\right\} &= \frac{c(T^{-1}\omega)}{\bar{c}(\omega)}, \end{aligned}\right.$$

and the initial distribution $d\nu(\omega) = \dfrac{\bar{c}(\omega)}{\int \bar{c} d\mu} d\mu(\omega)$. The transition operator of this chain is

$$\begin{aligned} P\psi(\omega) &= \mathbb{E}\left\{\psi(W_1)/W_0 = \omega\right\} \\ &= \psi(T\omega)\frac{c}{\bar{c}}(\omega) + \psi(T^{-1}\omega)\frac{c(T^{-1}\omega)}{\bar{c}(\omega)} \end{aligned} \tag{4.15}$$

with ψ be a bounded measurable function on Ω.

Lemma 4.2.1. $(W_n)_{n\geq 0}$ *is a stationary, ergodic Markov chain.*

Proof. One has

$$\begin{aligned} \mathbb{E}_\nu\left\{\psi(W_1)\right\} &= \mathbb{E}_\nu\left\{\mathbb{E}(\psi(W_1)/W_0)\right\} \\ &= \int \left(\psi \circ T \cdot \frac{c}{\bar{c}} + \psi \circ T^{-1}\frac{c(T^{-1})}{\bar{c}}\right)\frac{\bar{c}}{\int \bar{c} d\mu} d\mu \\ &= \int \psi \circ T \cdot c \frac{1}{\int \bar{c} d\mu} d\mu + \int \psi \circ T^{-1} \cdot c(T^{-1}) \frac{1}{\int \bar{c} d\mu} d\mu \\ &= \int \psi \cdot c(T^{-1}) \frac{1}{\int \bar{c} d\mu} d\mu + \int \psi \cdot c \frac{1}{\int \bar{c} d\mu} d\mu \\ &= \int \psi \left[c(T^{-1}) + c\right] \frac{1}{\int \bar{c} d\mu} d\mu \\ &= \int \psi d\nu. \end{aligned}$$

which shows that the chain is stationary.

For the ergodicity of the chain, we suppose $P\psi(\omega) = \psi(\omega)$, $\forall \omega \in \Omega$ then

$$\psi(\omega) = \psi(T\omega)\frac{c}{\bar{c}}(\omega) + \psi(T^{-1}\omega)\frac{c(T^{-1}\omega)}{\bar{c}(\omega)}. \tag{4.16}$$

We claim that ψ is a constant. Put

$$Q(\omega) = \int \sum_{y\in\Lambda} \bar{c}(\omega)p(\omega; y)\left[\psi(T^y\omega) - \psi(\omega)\right]^2 d\nu$$

then $Q(\omega) = 0$. Indeed, we have

$$\begin{aligned} Q(\omega) &= \int \sum_{y\in\Lambda} \bar{c}(\omega)p(\omega; y)\psi^2(T^y\omega)d\nu - 2\int \sum_{y\in\Lambda} \bar{c}(\omega)p(\omega; y)\psi(T^y\omega)\psi(\omega)d\nu \\ &\quad + \int \sum_{y\in\Lambda} \bar{c}(\omega)p(\omega; y)\psi^2(\omega)d\nu \\ &= \int \sum_{y\in\Lambda} \bar{c}(T^{-y}\omega)p(T^{-y}\omega; y)\psi^2(\omega)d\nu - 2\int \bar{c}(\omega)\psi^2(\omega)d\nu + \int \bar{c}(\omega)\psi^2(\omega)d\nu \end{aligned}$$

Since

$$\sum_{y\in\Lambda} \bar{c}(T^{-y}\omega)p(T^{-y}\omega; y) = \bar{c}(T^{-1}\omega)p(T^{-1}\omega; 1) + \bar{c}(T\omega)p(T\omega; -1)$$

$$\begin{aligned} &= \bar{c}(T^{-1}\omega)\frac{c(T^{-1}\omega)}{\bar{c}(T^{-1}\omega)} + \bar{c}(T\omega)\frac{c(\omega)}{\bar{c}(T\omega)} = c(T^{-1}\omega) + c(\omega) \\ &= \bar{c}(\omega) \end{aligned}$$

then $Q(\omega) = 0$. By the hypothesis $c > 0$, one obtains $\psi(T^y\omega) = \psi(\omega)$ ν *a.e.* And, by the ergodicity of T^y, $y \neq 0$ then ψ is a constant. Hence, $(W_n)_{n\geq 0}$ is ergodic. We have thus proved that $(W_n)_{n\geq 0}$ is a stationary ergodic Markov chain in Ω. $\square$

Therefore, the formula (4.13) can be written as

$$\begin{aligned} \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_\omega\left\{Y_i^2/\mathcal{F}_{i-1}\right\} &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\varphi(W_{i-1}) = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\varphi\pi_0(\sigma^{i-1}) \\ &= \int(\varphi\circ\pi_0)d\mathbb{P}_\nu, \quad \mathbb{P}_\nu \ a.e. \end{aligned}$$

by Birkhoff's ergodic theorem since W_n is ergodic. And then,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_\omega\left\{Y_i^2/\mathcal{F}_{i-1}\right\} = \int\varphi d\nu, \ \nu \ a.e \ \omega.$$

Moreover,

$$\begin{aligned} \lim_{n\to\infty}\frac{s_n^2}{n} &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_\omega\left\{Y_i^2\right\} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_\omega\left\{\varphi(W_{i-1})\right\} \\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}P^i\varphi(\omega) = \int\varphi d\nu \end{aligned} \tag{4.17}$$

by Hopf's ergodic theorem. Therefore,

$$I_1 = \lim_{n\to\infty}\frac{\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_\omega\left\{Y_i^2/\mathcal{F}_{i-1}\right\}}{\frac{1}{n}s_n^2} = 1, \ \nu \ a.e. \ \omega$$

which completes the proof of (4.11).

$\square$

Proof of (4.12). Fix $M > 0$, one has

$$\begin{aligned} &\mathbb{E}_\omega\left\{Y_i^2\mathbb{1}_{\{|Y_i|>\delta M\}}/X_{i-1}=k\right\} \\ =& \left(\frac{1}{c(T^k\omega)}\right)^2\mathbb{1}_{\left\{\frac{1}{c(T^k\omega)}>\delta M\right\}}\frac{c(T^k\omega)}{\bar{c}(T^k\omega)} + \left(\frac{1}{c(T^{k-1}\omega)}\right)^2\mathbb{1}_{\left\{\frac{1}{c(T^{k-1}\omega)}>\delta M\right\}}\frac{c(T^{k-1}\omega)}{\bar{c}(T^k\omega)} \\ =& \left(\frac{1}{c(T^k\omega)}\mathbb{1}_{\left\{\frac{1}{c(T^k\omega)}>\delta M\right\}} + \frac{1}{c(T^{k-1}\omega)}\mathbb{1}_{\left\{\frac{1}{c(T^{k-1}\omega)}>\delta M\right\}}\right)\frac{1}{\bar{c}(T^k\omega)} \\ =& \varphi_M(T^k\omega) \end{aligned}$$

where $\varphi_M = \left(\frac{1}{c}\mathbb{1}_{\left\{\frac{1}{c}>\delta M\right\}} + \frac{1}{c(T^{-1})}\mathbb{1}_{\left\{\frac{1}{c(T^{-1})}>\delta M\right\}}\right)\frac{1}{\bar{c}}$. Hence,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_\omega\left\{Y_i^2\mathbb{1}_{\{|Y_i|>\delta M\}}/\mathcal{F}_{i-1}\right\} = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\varphi_M\left(T^{X_{i-1}}\omega\right) = \int\varphi_M d\nu,$$

ν *a.e.* ω by the above similar way. Since $\lim_{n\to\infty} s_n = +\infty$, there exists $N > 0$ such that for any $n > N$ then $s_n > M$. And therefore,

$$\frac{1}{s_n^2}\sum_{i=N}^{n}\mathbb{E}_\omega\left\{Y_i^2\mathbb{1}_{\{|Y_i|>\delta s_n\}}/\mathcal{F}_{i-1}\right\} \le \frac{1}{s_n^2}\sum_{i=N}^{n}\mathbb{E}_\omega\left\{Y_i^2\mathbb{1}_{\{|Y_i|>\delta M\}}/\mathcal{F}_{i-1}\right\}$$

for any $n > N$. For $n\to\infty$, we obtain

$$I_2 \quad\le\quad \frac{\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_\omega\left\{Y_i^2\mathbb{1}_{\{|Y_i|>\delta M\}}/\mathcal{F}_{i-1}\right\}}{\lim_{n\to\infty}\frac{s_n^2}{n}} = \int\varphi_M d\nu.$$

Since $\int\varphi_M d\nu \longrightarrow 0 \ as \ M \longrightarrow +\infty$, then $I_2 = 0$ which completes the proof of (4.12).

□

The proposition 4.2.1 is then followed since by (4.17) one has

$$\begin{aligned}\lim_{n\to\infty}\frac{s_n^2}{n} &= \int\varphi\, d\nu = \int\left(\frac{1}{c}+\frac{1}{c(T^{-1})}\right)\frac{1}{\bar{c}}d\nu \\ &= \int\left(\frac{1}{c}+\frac{1}{c(T^{-1})}\right)\frac{1}{\int\bar{c}d\mu}d\mu \\ &= \frac{\int\frac{1}{c}d\mu}{\int c\, d\mu}.\end{aligned}$$

□

Proposition 4.2.2. $\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(X_n - \frac{1}{\int\frac{1}{c}d\mu}F(\omega,X_n)\right) = 0 \ in \ L^2.$

Proof. By the pointwise ergodic theorem

$$\lim_{h\to\infty}\frac{F(\omega,h)}{h} = \int\frac{1}{c}d\mu,\ \mu\ a.e.$$

then for any ε such that $0<\varepsilon<\int\frac{1}{c}d\mu$, there exists $M(\varepsilon)>0$ such that for any $|X_n| > M(\varepsilon)$ we have

$$\left|\frac{F(\omega,X_n)}{X_n} - \int\frac{1}{c}d\mu\right| < \varepsilon.$$

It follows that

$$|X_n| - \left|\frac{F(\omega, X_n)}{\int \frac{1}{c} d\mu}\right| < \left|\frac{F(\omega, X_n)}{\int \frac{1}{c} d\mu} - X_n\right| < \frac{\varepsilon}{\int \frac{1}{c} d\mu}|X_n| \tag{4.18}$$

and hence,

$$|X_n|\left(1 - \frac{\varepsilon}{\int \frac{1}{c} d\mu}\right) < \left|\frac{F(\omega, X_n)}{\int \frac{1}{c} d\mu}\right|$$

implies that

$$\frac{\varepsilon}{\int \frac{1}{c} d\mu}|X_n| < \left(\frac{\varepsilon}{\int \frac{1}{c} d\mu \cdot (\int \frac{1}{c} d\mu - \varepsilon)}\right)|F(\omega, X_n)|. \tag{4.19}$$

Combining (4.18) and (4.19), one has

$$\left|\frac{F(\omega, X_n)}{\int \frac{1}{c} d\mu} - X_n\right| < \frac{\varepsilon}{\int \frac{1}{c} d\mu}|X_n| < \left(\frac{\varepsilon}{\int \frac{1}{c} d\mu \cdot (\int \frac{1}{c} d\mu - \varepsilon)}\right)|F(\omega, X_n)|$$

for any $|X_n| > M(\varepsilon)$.

Put

$$H(\varepsilon) = \sup_{|h| \le M(\varepsilon)} \frac{|F(\omega, h)|}{\int \frac{1}{c} d\mu}.$$

One has

$$\begin{aligned}
\left|\frac{F(\omega, X_n)}{\int \frac{1}{c} d\mu} - X_n\right| &\le \max\left\{\frac{\varepsilon}{\int \frac{1}{c} d\mu}|X_n|; M(\varepsilon) + H(\varepsilon)\right\} \\
&\le \max\left\{\left(\frac{\varepsilon}{\int \frac{1}{c} d\mu \cdot (\int \frac{1}{c} d\mu - \varepsilon)}\right)|F(\omega, X_n)|; M(\varepsilon) + H(\varepsilon)\right\} \\
&\le \max\left\{\left(\frac{\varepsilon}{\int \frac{1}{c} d\mu \cdot (\int \frac{1}{c} d\mu - \varepsilon)}\right)|M_n|; M(\varepsilon) + H(\varepsilon)\right\}
\end{aligned}$$

and hence if we put $N_n = \frac{1}{\sqrt{n}}\left(\frac{F(\omega, X_n)}{\int \frac{1}{c} d\mu} - X_n\right)$ then

$$\begin{aligned}
N_n^2 &\le \max\left\{\left(\frac{\varepsilon}{\int \frac{1}{c} d\mu \cdot (\int \frac{1}{c} d\mu - \varepsilon)} \cdot \frac{M_n}{\sqrt{n}}\right)^2; \frac{[M(\varepsilon) + H(\varepsilon)]^2}{n}\right\} \\
&\le \left(\frac{\varepsilon}{\int \frac{1}{c} d\mu \cdot (\int \frac{1}{c} d\mu - \varepsilon)}\right)^2 \left(\frac{M_n}{\sqrt{n}}\right)^2 + \frac{[M(\varepsilon) + H(\varepsilon)]^2}{n}.
\end{aligned}$$

Therefore,

$$\mathbb{E}_\omega\left\{N_n^2\right\} \le \left(\frac{\varepsilon}{\int \frac{1}{c} d\mu \cdot (\int \frac{1}{c} d\mu - \varepsilon)}\right)^2 \mathbb{E}_\omega\left\{\left(\frac{M_n}{\sqrt{n}}\right)^2\right\} + \frac{\mathbb{E}_\omega\left\{[M(\varepsilon) + H(\varepsilon)]^2\right\}}{n}.$$

Since

$$\lim_{n\to\infty} \mathbb{E}_\omega \left\{ \left(\frac{M_n}{\sqrt{n}} \right)^2 \right\} = \frac{\int \frac{1}{c} d\mu}{\int c \, d\mu} < +\infty$$

and ε as small as we need, we have $\lim_{n\to\infty} \mathbb{E}_\omega \left\{ N_n^2 \right\} = 0$. We finished the proof of proposition 4.2.2 and theorem 4.2.1 is then followed. $\square$

Chapter 5

Central limit theorem for reversible Random walk in Random environment based on moments and analogue for continuous time

The main aim of this chapter is to introduce a new way to obtain again the Quenched CLT for reversible Random walk in Random environment in the preceding chapter without using any martingale. More precisely, for a given realization ω of the environment, we consider Poisson's equation $(P_\omega - I)g = f$ and then use the pointwise ergodic theorem to treat the limit of the solutions, the CLT will be establish by the convergence of the moments. In particular, there is an analogue for Markov process with continuous time and discrete space.

5.1 Random walk in random environment

Consider, on the $\mathbb{Z}$ network, a random stationary sequence of conductances, defined on a probability space $(\Omega, \mathcal{A}, \mu)$, an invertible $\mu-$preserving transformation T which is also ergodic, and a random variable $c > 0$. The space Ω is interpreted as the space of environments.

For a fixed environment $\omega \in \Omega$, the conductances of the edges $[k, k+1]$ is $c(T^k\omega)$ and $[k, k-1]$ is $c(T^{k-1}\omega)$.

Let $\bar{c} = c + c \circ T^{-1}$. We introduce the random walk $(X_n)_{n\geq 0}$ on $\mathbb{Z}$ with initial condition $X_0 = 0$ and Markov 's operator $f \longmapsto P_\omega f$ defined by

$$P_\omega f(k) = \frac{1}{\bar{c}(T^k\omega)} \Big[c(T^{k-1}\omega) f(k-1) + c(T^k\omega) f(k+1) \Big] . \tag{5.1}$$

In the sequel of this section, theorem 5.1.1, we will establish a Quenched central limit theorem for random walk $(X_n)_{n\geq 0}$. The method is to use the pointwise ergodic theorem and without using any martingale.

Theorem 5.1.1. *For almost all environment ω,*

$$\frac{X_n}{\sqrt{n}} \xrightarrow{\ D\ } \mathcal{N}\left(0, \sigma^2\right) \quad as \ \ n \longrightarrow +\infty$$

if c and $c^{-1} \in L^1(\mu)$ and where $\sigma^2 = \left[\int \frac{1}{c}\, d\mu \int c\, d\mu\right]^{-1}$.

***Remark* 5.1.1.** *If c or $c^{-1} \notin L^{-1}(\mu)$ then $\frac{X_n}{\sqrt{n}} \longrightarrow 0$ as $n \longrightarrow +\infty$ (Depauw and Derrien [12]).*

Consider a standard normal distribution $Z \sim \mathcal{N}(0,1)$, for each $\ell = 1, 2, 3, \ldots$, one has

$$\mathbb{E}\left\{Z^\ell\right\} = \begin{cases} 0 & if \quad \ell = 2k-1 \\ \frac{(2k)!}{k!2^k} & if \quad \ell = 2k \end{cases}$$

By the method of moments which was introduced in [3] (Billingsley's book: "Probability and measure", theorem 30.2, page 390), to prove theorem 5.1.1 we have to show that for almost all environment ω

$$\lim_{n\to+\infty} \mathbb{E}_\omega\left\{\left(\frac{X_n}{\sigma\sqrt{n}}\right)^\ell\right\} = \mathbb{E}\left\{Z^\ell\right\} = \begin{cases} 0 & if \quad \ell = 2k-1 \\ \frac{(2k)!}{k!2^k} & if \quad \ell = 2k \end{cases}$$

for each $\ell = 1, 2, 3, \ldots$. In the sequel, we will use the pointwise ergodic theorem to treat these limits. It is adapted from Depauw and Derrien [12].

Theorem 5.1.2. *(Depauw and Derrien, [12]) For almost all environments ω, we have*

$$\lim_{n\to+\infty} \mathbb{E}_\omega\left\{\frac{X_n^2}{n}\right\} = \sigma^2. \tag{5.2}$$

Proof. Fix $\omega \in \Omega$. We consider a function $f_1 \geq 0$, defined on $\mathbb{Z}$, such that $(P_\omega - I)f_1 \equiv 1$ and $f_1(0) = 0$. For example, we can take

$$f_1(m) = \begin{cases} \sum\limits_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)} \sum\limits_{s=1}^{\ell} \bar{c}(T^s\omega), & if \quad m \geq 1 \\ 0, & if \quad m = 0 \\ \sum\limits_{\ell=1}^{-m} \frac{1}{c(T^{-\ell}\omega)} \sum\limits_{s=1}^{\ell-1} \bar{c}(T^{-s}\omega), & if \quad m \leq -1 \end{cases}$$

It is easy to check that the function f_1 satisfies

$$(P_\omega - I)f_1(m) = 1, \quad \forall m \in \mathbb{Z}.$$

Replace m by X_n and take the expectation

$$\mathbb{E}_\omega\left\{(P_\omega - I)f_1(X_n)\right\} = 1, \quad \forall n \geq 0.$$

This is equivalent to

$$\mathbb{E}_\omega\left\{f_1(X_{n+1})\right\} - \mathbb{E}_\omega\left\{f_1(X_n)\right\} = 1, \quad \forall n \geq 0.$$

Since $\mathbb{E}_\omega\{f_1(X_0)\} = \mathbb{E}_\omega\{f_1(0)\} = 0$, we will obtain

$$\mathbb{E}_\omega\{f_1(X_n)\} = n, \quad \forall n \geq 0. \tag{5.3}$$

The formula (5.3) can be rewritten as

$$\mathbb{E}_\omega\left\{\frac{f_1(X_n)}{X_n^2} \times \frac{X_n^2}{n}\right\} = 1$$

and note that if $\lim_{m\to\infty}\frac{f_1(m)}{m^2}$ exists then so $\lim_{n\to+\infty}\mathbb{E}_\omega\left\{\frac{X_n^2}{n}\right\}$.

The next step we will compute the limit of $\frac{f_1(m)}{m^2}$ by using the pointwise ergodic theorem. We need the following lemma in the proof:

Lemma 5.1.1. *Let u_n and v_n be two sequences of positive real numbers and let U_n be a partial sum $U_n = \sum_{\ell=1}^n u_\ell$. Assume that*

$$\lim_{n\to\infty}\frac{1}{n}U_n = u \quad \text{and} \quad \lim_{n\to\infty} v_n = v \tag{5.4}$$

then for each $\alpha \in \mathbb{N}$

$$\lim_{n\to\infty}\frac{1}{n^{\alpha+1}}\sum_{\ell=1}^n \ell^\alpha u_\ell v_\ell = \frac{uv}{\alpha+1}. \tag{5.5}$$

Proof. Firstly consider the case $\alpha = 0$, we will show that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\ell=1}^n u_\ell v_\ell = uv. \tag{5.6}$$

One has

$$\begin{aligned}\left|\frac{1}{n}\sum_{\ell=1}^n u_\ell v_\ell - uv\right| &\leq \left|\frac{1}{n}\sum_{\ell=1}^n u_\ell(v_\ell - v)\right| + \left|\frac{1}{n}\sum_{\ell=1}^n (u_\ell - u)v\right| \\ &\leq \frac{1}{n}\sum_{\ell=1}^n u_\ell\left|v_\ell - v\right| + v\left|\frac{1}{n}\sum_{\ell=1}^n u_\ell - u\right| \\ &< \varepsilon\end{aligned}$$

for any $\varepsilon > 0$ when n large enough which completes (5.6).

Now assume that (5.5) is true for $\alpha \geq 0$, we claim that it holds also for $\alpha + 1$ that is

$$\lim_{n\to\infty}\frac{1}{n^{\alpha+2}}\sum_{\ell=1}^n \ell^{\alpha+1} u_\ell v_\ell = \frac{uv}{\alpha+2}. \tag{5.7}$$

Put $W_n = \sum_{\ell=1}^n \ell^\alpha u_\ell v_\ell$, using Abel's transformation

$$\begin{aligned}\frac{1}{n^{\alpha+2}}\sum_{\ell=1}^n \ell^{\alpha+1}u_\ell v_\ell &= -\frac{1}{n^{\alpha+2}}\sum_{\ell=1}^{n-1} W_\ell + \frac{1}{n^{\alpha+1}}W_n \\ &= -I_1 + I_2.\end{aligned}$$

By the assumption $\lim_{n\to\infty} I_2 = \lim_{n\to\infty} \frac{1}{n^{\alpha+1}} W_n = \frac{uv}{\alpha+1}$, and one has

$$\begin{aligned}\left|I_1 - \frac{uv}{(\alpha+1)(\alpha+2)}\right| &\leq \frac{1}{n^{\alpha+2}}\sum_{\ell=1}^{n-1}\ell^{\alpha+1}\left|\frac{W_\ell}{\ell^{\alpha+1}} - \frac{uv}{\alpha+1}\right| + \left|\frac{1}{n^{\alpha+2}}\sum_{\ell=1}^{n-1}\ell^{\alpha+1} - \frac{1}{\alpha+2}\right|\frac{uv}{\alpha+1}\\ &< \varepsilon\end{aligned}$$

for any $\varepsilon > 0$ when n large enough since $\lim_{n\to\infty} \frac{1}{n^{\alpha+2}} \sum_{\ell=1}^{n-1} \ell^{\alpha+1} = \int_0^1 x^{\alpha+1} dx = \frac{1}{\alpha+2}$. It follows that $\lim_{n\to\infty} I_1 = \frac{uv}{(\alpha+1)(\alpha+2)}$. And hence,

$$\lim_{n\to\infty} \frac{1}{n^{\alpha+2}} \sum_{\ell=1}^{n} \ell^{\alpha+1} u_\ell v_\ell = -\frac{uv}{(\alpha+1)(\alpha+2)} + \frac{uv}{\alpha+1} = \frac{uv}{\alpha+2}$$

which completes (5.7). □

Lemma 5.1.2. *With f_1 defined as above, we have*

$$\lim_{m\to\pm\infty} \frac{f_1(m)}{m^2} = \int_\Omega \frac{1}{c}\, d\mu \int_\Omega c\, d\mu = \sigma^{-2}. \tag{5.8}$$

Proof. Consider firstly the case $m > 0$. Applying lemma 5.1.1 for $u_\ell = \frac{1}{c(T^\ell\omega)}$, $v_\ell = \frac{1}{\ell}\sum_{s=1}^{\ell} \bar{c}(T^s\omega)$ and $\alpha = 1$, one has

$$\begin{aligned}\lim_{m\to+\infty} \frac{f_1(m)}{m^2} &= \lim_{m\to+\infty} \frac{1}{m^2} \sum_{\ell=0}^{m-1} \frac{\ell}{c(T^\ell\omega)} \frac{1}{\ell} \sum_{s=1}^{\ell} \bar{c}(T^s\omega) = \frac{1}{2}\int_\Omega \frac{1}{c}\, d\mu \int_\Omega \bar{c}\, d\mu\\ &= \int_\Omega \frac{1}{c}\, d\mu \int_\Omega c\, d\mu.\end{aligned}$$

Similarly, one has the same result for the case $m < 0$. □

From lemma 5.1.2, for any $\varepsilon > 0$, there exists $M > 0$ such that for any $m > M$ then

$$\left|\frac{m^2}{f_1(m)} - \sigma^2\right| < \varepsilon/2. \tag{5.9}$$

Now we combine (5.3) and (5.9) to prove theorem 5.1.2. Put

$$K_1 = \mathbb{E}_\omega\left\{\frac{X_n^2}{n}\mathbb{1}_{\{|X_n|>M\}}\right\} - \mathbb{E}_\omega\left\{\sigma^2\frac{f_1(X_n)}{n}\mathbb{1}_{\{|X_n|>M\}}\right\},$$

$$K_2 = \mathbb{E}_\omega\left\{\frac{X_n^2}{n}\mathbb{1}_{\{|X_n|\leq M\}}\right\} - \mathbb{E}_\omega\left\{\sigma^2\frac{f_1(X_n)}{n}\mathbb{1}_{\{|X_n|\leq M\}}\right\}$$

For n large enough

$$\begin{aligned}|K_1| &= \left|\mathbb{E}_\omega\left\{\left(\frac{X_n^2}{f_1(X_n)} - \sigma^2\right)\frac{f_1(X_n)}{n}\mathbb{1}_{\{|X_n|>M\}}\right\}\right|\\ &\leq \mathbb{E}_\omega\left\{\left|\frac{X_n^2}{f_1(X_n)} - \sigma^2\right|\frac{f_1(X_n)}{n}\mathbb{1}_{\{|X_n|>M\}}\right\} < \varepsilon/2\end{aligned}$$

since $f_1(m) \geq 0$ for any $m \in \mathbb{Z}$, and

$$\begin{aligned} |K_2| &= \left| \frac{1}{n} \mathbb{E}_\omega \left\{ \left(X_n^2 - \sigma^2 f_1(X_n) \right) \mathbb{1}_{\{|X_n| \leq M\}} \right\} \right| \\ &\leq \frac{1}{n} \mathbb{E}_\omega \left\{ \left| X_n^2 - \sigma^2 f_1(X_n) \right| \mathbb{1}_{\{|X_n| \leq M\}} \right\} < \varepsilon/2. \end{aligned}$$

It follows that

$$\left| \mathbb{E}_\omega \left\{ \frac{X_n^2}{n} \right\} - \sigma^2 \right| = |K_1 + K_2| \leq |K_1| + |K_2| < \varepsilon$$

for n large enough. Since ε is as small as we need, then $\lim_{n \to +\infty} \mathbb{E}_\omega \left\{ \frac{X_n^2}{n} \right\} = \sigma^2$.

□

Theorem 5.1.3. *For almost all environments ω, we have*

$$\lim_{n \to +\infty} \mathbb{E}_\omega \left\{ \left(\frac{X_n^2}{n} \right)^k \right\} = \frac{(2k)!}{2^k k!} \sigma^{2k} \tag{5.10}$$

for each $k \geq 1$.

This is the generalization of theorem 5.1.2.

Proof. We will use the similar method in theorem 5.1.2 to prove theorem 5.1.3.

Fix $\omega \in \Omega$. We consider a sequence of functions $f_k \geq 0$, defined on $\mathbb{Z}$, such that $(P_\omega - I) f_{k+1} \equiv f_k$, $f_k(0) = 0$ and f_1 is defined as above. For instance, we can take for $k \geq 1$

$$\begin{aligned} f_0(m) &= 1 \quad \forall m \in \mathbb{Z} \\ f_1(m) &= \begin{cases} \sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell \omega)} \sum_{s=1}^{\ell} \bar{c}(T^s \omega) f_0(s), & if \quad m \geq 1 \\ 0, & if \quad m = 0 \\ \sum_{\ell=1}^{-m} \frac{1}{c(T^{-\ell} \omega)} \sum_{s=1}^{\ell-1} \bar{c}(T^{-s} \omega) f_0(-s), & if \quad m \leq -1 \end{cases} \\ f_2(m) &= \begin{cases} \sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell \omega)} \sum_{s=1}^{\ell} \bar{c}(T^s \omega) f_1(s), & if \quad m \geq 1 \\ 0, & if \quad m = 0 \\ \sum_{\ell=1}^{-m} \frac{1}{c(T^{-\ell} \omega)} \sum_{s=1}^{\ell-1} \bar{c}(T^{-s} \omega) f_1(-s), & if \quad m \leq -1 \end{cases} \\ &\cdots \\ f_k(m) &= \begin{cases} \sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell \omega)} \sum_{s=1}^{\ell} \bar{c}(T^s \omega) f_{k-1}(s), & if \quad m \geq 1 \\ 0, & if \quad m = 0 \\ \sum_{\ell=1}^{-m} \frac{1}{c(T^{-\ell} \omega)} \sum_{s=1}^{\ell-1} \bar{c}(T^{-s} \omega) f_{k-1}(-s), & if \quad m \leq -1 \end{cases} \end{aligned}$$

It is easy to check that

$$(P_\omega - I) f_k(m) = f_{k-1}(m), \quad \forall m \in \mathbb{Z}.$$

Replace m by X_n and take the expectation

$$\mathbb{E}_\omega \left\{(P_\omega - I)f_k(X_n)\right\} = \mathbb{E}_\omega \left\{f_{k-1}(X_n)\right\}, \quad \forall n \geq 0.$$

It follows that

$$\mathbb{E}_\omega \left\{f_k(X_{n+1})\right\} = \mathbb{E}_\omega \left\{f_k(X_n)\right\} + \mathbb{E}_\omega \left\{f_{k-1}(X_n)\right\}, \quad \forall n \geq 0.$$

Lemma 5.1.3. *With n large enough and for each $k \geq 1$, then*

$$\mathbb{E}_\omega \left\{f_k(X_n)\right\} \sim \frac{n^k}{k!}. \tag{5.11}$$

Proof. It is obvious to work with $k = 1$.

Assume that it is true with $k \geq 1$, we claim that it is also with $k+1$. That means: if

$$\mathbb{E}_\omega \left\{f_k(X_n)\right\} \sim \frac{n^k}{k!}$$

then

$$\mathbb{E}_\omega \left\{f_{k+1}(X_n)\right\} \sim \frac{n^{k+1}}{(k+1)!}.$$

Since

$$\begin{aligned} \mathbb{E}_\omega \left\{f_{k+1}(X_n)\right\} &= \mathbb{E}_\omega \left\{f_{k+1}(X_{n-1})\right\} + \mathbb{E}_\omega \left\{f_k(X_{n-1})\right\} \quad \forall n \geq 1. \\ &\sim \sum_{i=1}^{n} \frac{(i-1)^k}{k!} \end{aligned}$$

with n large enough.

Using the fact

$$\sum_{i=1}^{n} i^k \sim \frac{1}{k+1} n^{k+1} \tag{5.12}$$

for each $k \geq 1$ when n large enough then

$$\mathbb{E}_\omega \left\{f_{k+1}(X_n)\right\} \sim \frac{(n-1)^{k+1}}{(k+1)!} \sim \frac{n^{k+1}}{(k+1)!}.$$

$\square$

The formula (5.11) can be rewritten as

$$\mathbb{E}_\omega \left\{\frac{f_k(X_n)}{X_n^{2k}} \times \frac{X_n^{2k}}{n^k}\right\} \sim \frac{1}{k!}$$

and note that if $\lim_{m\to\infty} \frac{f_k(m)}{m^{2k}}$ exists then so $\lim_{n\to+\infty} \mathbb{E}_\omega \left\{\frac{X_n^{2k}}{n^k}\right\}$.

The next step we will compute the limit of $\frac{f_1(m)}{m^2}$ by using the pointwise ergodic theorem and lemma 5.1.1.

Lemma 5.1.4. *For each $k \geq 1$, with f_k defined as above then*

$$\lim_{m\to\pm\infty} \frac{f_k(m)}{m^{2k}} = \frac{2^k}{(2k)!}\sigma^{-2k} = C_k. \tag{5.13}$$

Proof. This limit is true for $k = 1$ (lemma 5.1.2).

Assume that (5.13) is also true for $k \geq 1$, we claim that it holds for $k+1$, that is

$$\lim_{m\to\pm\infty} \frac{f_{k+1}(m)}{m^{2(k+1)}} = \frac{2^{k+1}}{(2(k+1))!}\sigma^{-2(k+1)}. \tag{5.14}$$

Consider firstly the case $m > 0$. Applying lemma 5.1.1 for $u_s = \bar{c}(T^s\omega)$, $v_s = \frac{1}{s^{2k}} f_k(s)$ and $\alpha = 2k$, one has

$$\begin{aligned}
\lim_{\ell\to+\infty} \frac{1}{\ell^{2k+1}} \sum_{s=1}^{\ell} \bar{c}(T^s\omega) f_k(s) &= \lim_{\ell\to+\infty} \frac{1}{\ell^{2k+1}} \sum_{s=1}^{\ell} s^{2k}\bar{c}(T^s\omega)\frac{1}{s^{2k}} f_k(s) \\
&= \int_\Omega c\, d\mu \frac{2^{k+1}}{(2k+1)!}\sigma^{-2k}.
\end{aligned}$$

Again, applying lemma 5.1.1 for $u'_\ell = \frac{1}{c(T^\ell\omega)}$, $v'_\ell = \frac{1}{\ell^{2k+1}} \sum_{s=1}^{\ell} \bar{c}(T^s\omega) f_k(s)$ and $\alpha = 2k+1$, one has

$$\begin{aligned}
\lim_{m\to+\infty} \frac{f_{2k+1}(m)}{m^{2(k+1)}} &= \lim_{m\to+\infty} \frac{1}{m^{2(k+1)}} \sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)} \sum_{s=1}^{\ell} \bar{c}(T^s\omega) f_k(s) \\
&= \lim_{m\to+\infty} \frac{1}{m^{2(k+1)}} \sum_{\ell=0}^{m-1} \frac{\ell^{2k+1}}{c(T^\ell\omega)} \frac{1}{\ell^{2k+1}} \sum_{s=1}^{\ell} \bar{c}(T^s\omega) f_k(s) \\
&= \frac{2^{k+1}}{(2(k+1))!}\sigma^{-2(k+1)}.
\end{aligned}$$

Similarly, one has the same result for the case $m < 0$. □

From lemma 5.1.4, for any $\varepsilon > 0$, there exists $M > 0$ such that for any $m > M$ then

$$\left| \frac{m^{2k}}{f_k(m)} - \frac{1}{C_k} \right| < \varepsilon/2. \tag{5.15}$$

Now we combine (5.11) and (5.15) to prove theorem 5.1.3. Put

$$K_3 = \mathbb{E}_\omega \left\{ \frac{X_n^{2k}}{n^k} \mathbb{1}_{\{|X_n|>M\}} \right\} - \mathbb{E}_\omega \left\{ \frac{f_k(X_n)}{n^k C_k} \mathbb{1}_{\{|X_n|>M\}} \right\}$$

$$K_4 = \mathbb{E}_\omega \left\{ \frac{X_n^{2k}}{n^k} \mathbb{1}_{\{|X_n|\leq M\}} \right\} - \mathbb{E}_\omega \left\{ \frac{f_k(X_n)}{n^k C_k} \mathbb{1}_{\{|X_n|\leq M\}} \right\}.$$

By lemma 5.1.3 and lemma 5.1.4, when n large enough, we have

$$|K_3| = \left| \mathbb{E}_\omega \left\{ \left(\frac{X_n^{2k}}{f_k(X_n)} - \frac{1}{C_k} \right) \frac{f_k(X_n)}{n^k} \mathbb{1}_{\{|X_n|>M\}} \right\} \right|$$

$$
\begin{aligned}
&\le \mathbb{E}_\omega \left\{ \left| \frac{X_n^{2k}}{f_k(X_n)} - \frac{1}{C_k} \right| \frac{f_k(X_n)}{n^k} \mathbb{1}_{\{|X_n|>M\}} \right\} \\
&< \varepsilon/2
\end{aligned}
$$

since function $f_k \ge 0$, and

$$
\begin{aligned}
|K_4| &= \left| \frac{1}{n^k} \mathbb{E}_\omega \left\{ \left(X_n^{2k} - \frac{1}{C_k} f_k(X_n) \right) \mathbb{1}_{\{|X_n|\le M\}} \right\} \right| \\
&\le \frac{1}{n^k} \mathbb{E}_\omega \left\{ \left| X_n^{2k} - \frac{1}{C_k} f_k(X_n) \right| \mathbb{1}_{\{|X_n|\le M\}} \right\} < \varepsilon/2.
\end{aligned}
$$

It follows that

$$
\left| \mathbb{E}_\omega \left\{ \left(\frac{X_n^2}{n} \right)^k \right\} - \frac{1}{k! C_k} \right| \approx |K_3 + K_4| \le |K_3| + |K_4| < \varepsilon
$$

for n large enough. Since ε is as small as we need, then we obtain the result

$$
\lim_{n \to +\infty} \mathbb{E}_\omega \left\{ \left(\frac{X_n^2}{n} \right)^k \right\} = \frac{(2k)!}{k! 2^k} \sigma^{2k}.
$$

□

Theorem 5.1.4. *For almost all environments ω, we have*

$$
\lim_{n \to +\infty} \mathbb{E}_\omega \left\{ \frac{X_n}{\sqrt{n}} \right\} = 0. \tag{5.16}
$$

Proof. Fix $\omega \in \Omega$.

We consider a function g_1, defined on $\mathbb{Z}$, satisfying $(P_\omega - I) g_1 \equiv 0$ and $g_1(0) = 0$. For instance, we can take

$$
g_1(m) = \begin{cases} \sum\limits_{\ell=0}^{m-1} \frac{1}{c(T^\ell \omega)}, & if \quad m \ge 1 \\ 0, & if \quad m = 0 \\ -\sum\limits_{\ell=1}^{-m} \frac{1}{c(T^{-\ell}\omega)}, & if \quad m \le -1 \end{cases}
$$

It is easy to check that

$$
(P_\omega - I) g_1(m) = 0, \quad \forall m \in \mathbb{Z}
$$

then

$$
(P_\omega - I) g_1(X_n) = 0, \quad \forall n \ge 0
$$

and take the expectation

$$
\mathbb{E}_\omega \{ P_\omega g_1(X_n) \} - \mathbb{E}_\omega \{ g_1(X_n) \} = 0, \quad \forall n \ge 0,
$$

and so

$$
\mathbb{E}_\omega \{ g_1(X_{n+1}) \} - \mathbb{E}_\omega \{ g_1(X_n) \} = 0, \quad \forall n \ge 0.
$$

It follows that

$$\mathbb{E}_\omega\left\{g_1(X_n)\right\} = \mathbb{E}_\omega\left\{g_1(X_0)\right\} = 0, \quad \forall n \geq 0. \tag{5.17}$$

The formula (5.17) can be rewritten as

$$\mathbb{E}_\omega\left\{\frac{g_1(X_n)}{X_n} \times \frac{X_n}{\sqrt{n}}\right\} = 0$$

and note that if $\lim_{m\to\infty} \frac{g_1(m)}{m}$ exists then so $\lim_{n\to+\infty} \mathbb{E}_\omega\left\{\frac{X_n}{\sqrt{n}}\right\}$.

The pointwise ergodic theorem ensures that

$$\lim_{m\to\infty} \frac{g_1(m)}{m} = \int_\Omega \frac{1}{c}\, d\mu = D_1. \tag{5.18}$$

Therefore, for any $\varepsilon > 0$, there exists $M > 0$ such that for any $|m| > M$ then

$$\left|\frac{g_1(m)}{D_1 m} - 1\right| < \varepsilon. \tag{5.19}$$

Now we combine (5.17) and (5.19) to prove theorem 5.1.4. Put

$$K_5 = \mathbb{E}_\omega\left\{\frac{1}{\sqrt{n}} X_n \mathbb{1}_{\{|X_n|\leq M\}}\right\} - \mathbb{E}_\omega\left\{\frac{1}{\sqrt{n}} \frac{g_1(X_n)}{D_1} \mathbb{1}_{\{|X_n|\leq M\}}\right\}$$

$$K_6 = \mathbb{E}_\omega\left\{\frac{1}{\sqrt{n}} X_n \mathbb{1}_{\{|X_n|> M\}}\right\} - \mathbb{E}_\omega\left\{\frac{1}{\sqrt{n}} \frac{g_1(X_n)}{D_1} \mathbb{1}_{\{|X_n|> M\}}\right\}.$$

For n large enough we have

$$|K_5| = \left|\mathbb{E}_\omega\left\{\frac{1}{\sqrt{n}}\left(X_n - \frac{g_1(X_n)}{D_1}\right)\mathbb{1}_{\{|X_n|\leq M\}}\right\}\right| < \varepsilon$$

and

$$\begin{aligned} |K_6| &= \left|\mathbb{E}_\omega\left\{\frac{1}{\sqrt{n}}\left(X_n - \frac{g_1(X_n)}{D_1}\right)\mathbb{1}_{\{|X_n|> M\}}\right\}\right| \\ &= \left|\mathbb{E}_\omega\left\{\frac{1}{\sqrt{n}}\left(1 - \frac{g_1(X_n)}{X_n D_1}\right)X_n\mathbb{1}_{\{|X_n|> M\}}\right\}\right| \\ &\leq \varepsilon \mathbb{E}_\omega\left\{\frac{|X_n|}{\sqrt{n}}\right\} \leq \varepsilon\sqrt{\mathbb{E}_\omega\left\{\frac{X_n^2}{n}\right\}}. \end{aligned}$$

It follows that

$$\left|\mathbb{E}_\omega\left\{\frac{X_n}{\sqrt{n}}\right\}\right| = |K_5 + K_6| \leq |K_5| + |K_6| < \varepsilon + \varepsilon\sqrt{\mathbb{E}_\omega\left\{\frac{X_n^2}{n}\right\}}$$

for n large enough. By theorem 5.1.2

$$\lim_{n\to+\infty} \mathbb{E}_\omega\left\{\frac{X_n^2}{n}\right\} = \sigma^2 < \infty$$

and ε is as small as we need, then we obtain $\lim\limits_{n\to+\infty} \mathbb{E}_\omega\left\{\frac{X_n}{\sqrt{n}}\right\} = 0.$ □

Theorem 5.1.5. *For almost all environments ω, we have*

$$\lim_{n\to+\infty} \mathbb{E}_\omega \left\{ \left(\frac{X_n}{\sqrt{n}} \right)^{2k-1} \right\} = 0 \tag{5.20}$$

for each $k \geq 1$.

Proof. Fix $\omega \in \Omega$.

We consider a sequence of functions g_k, defined on $\mathbb{Z}$, satisfying $(P_\omega - I)g_{k+1} \equiv g_k$ for any $k \geq 1$ and g_1 is defined as above. For instance, we can take for $k \geq 1$

$$g_{k+1}(m) = \begin{cases} \sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell \omega)} \sum_{s=1}^{\ell} \bar{c}(T^s\omega) g_k(s), & if \quad m \geq 1 \\ 0, & if \quad m = 0 \\ \sum_{\ell=1}^{-m} \frac{1}{c(T^{-\ell}\omega)} \sum_{s=1}^{\ell-1} \bar{c}(T^{-s}\omega) g_k(-s), & if \quad m \leq -1 \end{cases}$$

Then we have

$$(P_\omega - I)g_{k+1}(m) = g_k(m), \quad \forall m \in \mathbb{Z}.$$

Replace m by X_n and take the expectation

$$\mathbb{E}_\omega \left\{ P_\omega g_{k+1}(X_n) \right\} - \mathbb{E}_\omega \left\{ g_{k+1}(X_n) \right\} = \mathbb{E}_\omega \left\{ g_k(X_n) \right\}, \quad \forall n \geq 0$$

and so

$$\mathbb{E}_\omega \left\{ g_{k+1}(X_{n+1}) \right\} = \mathbb{E}_\omega \left\{ g_{k+1}(X_n) \right\} + \mathbb{E}_\omega \left\{ g_k(X_n) \right\}, \quad \forall n \geq 0.$$

Lemma 5.1.5. *With functions g_k defined as above*

$$\mathbb{E}_\omega \left\{ g_k(X_n) \right\} = 0, \quad \forall n \geq 0 \tag{5.21}$$

for each $k \geq 1$.

Proof. It is true with $k = 1$. Suppose it is also true with $k \geq 1$, that means

$$\mathbb{E}_\omega \left\{ g_k(X_n) \right\} = 0, \quad \forall n \geq 0$$

we want to show that

$$\mathbb{E}_\omega \left\{ g_{k+1}(X_n) \right\} = 0, \quad \forall n \geq 0.$$

We have

$$\begin{aligned} \mathbb{E}_\omega \left\{ g_{k+1}(X_{n+1}) \right\} &= \mathbb{E}_\omega \left\{ g_{k+1}(X_n) \right\} + \mathbb{E}_\omega \left\{ g_k(X_n) \right\} \\ &= \mathbb{E}_\omega \left\{ g_{k+1}(X_n) \right\} = \ldots = \mathbb{E}_\omega \left\{ g_{k+1}(X_0) \right\} = 0. \end{aligned}$$

□

The formula (5.21) can be rewritten as

$$\mathbb{E}_\omega \left\{ \frac{g_k(X_n)}{X_n^{2k-1}} \times \frac{X_n^{2k-1}}{(\sqrt{n})^{2k-1}} \right\} = 0$$

and note that if $\lim_{m\to\infty} \frac{g_k(m)}{m^{2k-1}}$ exists then so $\lim_{n\to+\infty} \mathbb{E}_\omega \left\{ \left(\frac{X_n}{\sqrt{n}} \right)^{2k-1} \right\}$.

The next step we will compute the limit of $\frac{g_k(m)}{m^{2k-1}}$ by using the pointwise ergodic theorem and lemma 5.1.1.

Lemma 5.1.6. *For each $k \geq 1$ and g_k defined as above, we have*

$$\lim_{m\to\infty} \frac{g_k(m)}{m^{2k-1}} = \frac{2^{k-1}}{(2k-1)!} \left[\int_\Omega \frac{1}{c}\, d\mu \right]^k \left[\int_\Omega c\, d\mu \right]^{k-1} = D_k. \tag{5.22}$$

Proof. This limit is true for $k = 1$ (5.18).

Assume that (5.22) is also true for $k \geq 1$, we claim that it holds for $k+1$, that is

$$\lim_{m\to+\infty} \frac{g_{k+1}(m)}{m^{2k+1}} = \frac{2^k}{(2k+1)!} \left[\int_\Omega \frac{1}{c}\, d\mu \right]^{k+1} \left[\int_\Omega c\, d\mu \right]^k. \tag{5.23}$$

Consider firstly the case $m > 0$. Applying lemma 5.1.1 for $u_s = \bar{c}(T^s\omega)$, $v_s = \frac{1}{s^{2k-1}} g_k(s)$ and $\alpha = 2k-1$, one has

$$\begin{aligned} \lim_{\ell\to+\infty} \frac{1}{\ell^{2k}} \sum_{s=1}^{\ell} \bar{c}(T^s\omega) g_k(s) &= \lim_{\ell\to+\infty} \frac{1}{\ell^{2k}} \sum_{s=1}^{\ell} s^{2k-1} \bar{c}(T^s\omega) \frac{1}{s^{2k-1}} g_k(s) \\ &= \frac{2^k}{(2k)!} \left[\int_\Omega \frac{1}{c}\, d\mu \right]^k \left[\int_\Omega c\, d\mu \right]^k. \end{aligned}$$

Again, applying lemma 5.1.1 for $u'_\ell = \frac{1}{c(T^\ell\omega)}$, $v'_\ell = \frac{1}{\ell^{2k}} \sum_{s=1}^{\ell} \bar{c}(T^s\omega) g_k(s)$ and $\alpha = 2k$, one has

$$\begin{aligned} \lim_{m\to+\infty} \frac{g_{k+1}(m)}{m^{2k+1}} &= \lim_{m\to+\infty} \frac{1}{m^{2k+1}} \sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)} \sum_{s=1}^{\ell} \bar{c}(T^s\omega) g_k(s) \\ &= \lim_{m\to+\infty} \frac{1}{m^{2k+1}} \sum_{\ell=0}^{m-1} \frac{\ell^{2k}}{c(T^\ell\omega)} \frac{1}{\ell^{2k}} \sum_{s=1}^{\ell} \bar{c}(T^s\omega) g_k(s) \\ &= \frac{2^k}{(2k+1)!} \left[\int_\Omega \frac{1}{c}\, d\mu \right]^{k+1} \left[\int_\Omega c\, d\mu \right]^k. \end{aligned}$$

Similarly, one has the same result for the case $m < 0$. □

From lemma 5.1.6, for any $\varepsilon > 0$, there exists $M > 0$ such that for any $|m| > M$ then

$$\left| \frac{g_k(m)}{m^{2k-1} D_k} - 1 \right| < \varepsilon. \tag{5.24}$$

Now we combine (5.21) and (5.24) to prove theorem 5.1.5. Put

$$K_7 = \mathbb{E}_\omega \left\{ \frac{1}{(\sqrt{n})^{2k-1}} X_n^{2k-1} \mathbb{1}_{\{|X_n| \le M\}} \right\} - \mathbb{E}_\omega \left\{ \frac{1}{(\sqrt{n})^{2k-1}} \frac{g_k(X_n)}{D_k} \mathbb{1}_{\{|X_n| \le M\}} \right\}$$

$$K_8 = \mathbb{E}_\omega \left\{ \frac{1}{(\sqrt{n})^{2k-1}} X_n^{2k-1} \mathbb{1}_{\{|X_n| > M\}} \right\} - \mathbb{E}_\omega \left\{ \frac{1}{(\sqrt{n})^{2k-1}} \frac{g_k(X_n)}{D_k} \mathbb{1}_{\{|X_n| > M\}} \right\}.$$

For n large enough we have

$$|K_7| = \left| \mathbb{E}_\omega \left\{ \frac{1}{(\sqrt{n})^{2k-1}} \left(X_n^{2k-1} - \frac{g_k(X_n)}{D_k} \right) \mathbb{1}_{\{|X_n| \le M\}} \right\} \right| < \varepsilon$$

and

$$\begin{aligned} |K_8| &= \left| \mathbb{E}_\omega \left\{ \frac{1}{(\sqrt{n})^{2k-1}} \left(X_n^{2k-1} - \frac{g_k(X_n)}{D_k} \right) \mathbb{1}_{\{|X_n| > M\}} \right\} \right| \\ &= \left| \mathbb{E}_\omega \left\{ \frac{1}{(\sqrt{n})^{2k-1}} \left(1 - \frac{g_k(X_n)}{X_n^{2k-1} D_k} \right) X_n^{2k-1} \mathbb{1}_{\{|X_n| > M\}} \right\} \right| \\ &< \varepsilon \mathbb{E}_\omega \left\{ \left(\frac{|X_n|}{\sqrt{n}} \right)^{2k-1} \right\} \le \varepsilon \sqrt{\mathbb{E}_\omega \left\{ \left(\frac{X_n^2}{n} \right)^{2k-1} \right\}}. \end{aligned}$$

It follows that

$$\left| \mathbb{E}_\omega \left\{ \left(\frac{X_n}{\sqrt{n}} \right)^{2k-1} \right\} \right| = |K_7 + K_8| \le |K_7| + |K_8| < \varepsilon + \varepsilon \sqrt{\mathbb{E}_\omega \left\{ \left(\frac{X_n^2}{n} \right)^{2k-1} \right\}}$$

for n large enough. By theorem 5.1.3

$$\lim_{n \to \infty} \mathbb{E}_\omega \left\{ \left(\frac{X_n^2}{n} \right)^{2k-1} \right\} = \frac{[2(2k-1)]!}{(2k-1)! 2^{2k-1}} \sigma^{2(2k-1)}$$

and ε is as small as we need, then we obtain the result $\lim_{n \to +\infty} \mathbb{E}_\omega \left\{ \left(\frac{X_n}{\sqrt{n}} \right)^{2k-1} \right\} = 0.$ □

Finally, for each $\ell = 1, 2, 3, \ldots$ we obtain

$$\lim_{n \to \infty} \mathbb{E}_\omega \left\{ \left(\frac{X_n}{\sqrt{n}} \right)^{\ell} \right\} = \begin{cases} 0 & if \quad \ell = 2k-1 \\ \frac{(2k)!}{2^k k!} \sigma^{2k} & if \quad \ell = 2k \end{cases}$$

And hence, for almost all environment ω

$$\frac{X_n}{\sqrt{n}} \xrightarrow{\quad D \quad} \mathcal{N}\left(0, \sigma^2\right) \quad as\ n \longrightarrow +\infty$$

which completes the proof of theorem 5.1.1.

5.2 Markov process with discrete space

We consider Markov process $(X_t)_{t\in\mathbb{R}}$ on $\mathbb{Z}$ with $X_0 = 0$, the generator infinitesimal

$$L_\omega f(k) = c(T^{k-1}\omega)f(k-1) + c(T^k\omega)f(k+1) - \bar{c}(T^k\omega)f(k), \tag{5.25}$$

In the sequel of this section, theorem 5.2.1, we will establish a central limit theorem for Markov process $(X_t)_{t\in\mathbb{R}}$. We will use also an analogue method in section 5.1.

Theorem 5.2.1. *For almost environment ω,*

$$\frac{X_t}{\sqrt{t}} \xrightarrow{D} \mathcal{N}\left(0,\sigma^2\right) \quad as \;\; t \longrightarrow +\infty$$

if $c^{-1} \in L^1(\mu)$ and where $\sigma^2 = 2\left[\int \frac{1}{c}\, d\mu\right]^{-1}$.

Proof. As in theorem 5.1.1, to prove theorem 5.2.1 we have to show that for almost all environment ω

$$\lim_{t\to+\infty} \mathbb{E}_\omega\left\{\left(\frac{X_t}{\sigma\sqrt{t}}\right)^\ell\right\} = \begin{cases} 0 & if \quad \ell = 2k-1 \\ \frac{(2k)!}{k!2^k} & if \quad \ell = 2k \end{cases}$$

for each $\ell = 1, 2, 3, \ldots$. In the sequel, we will use also the pointwise ergodic theorem to treat these limits.

Theorem 5.2.2. *For almost all environments ω, we have*

$$\lim_{n\to+\infty} \mathbb{E}_\omega\left\{\frac{X_t^2}{t}\right\} = \sigma^2. \tag{5.26}$$

Proof. Fix $\omega \in \Omega$.

We consider a function $f_1 \geq 0$, defined on $\mathbb{Z}$, such that $L_\omega f_1 \equiv 1$ and $f_1(0) = 0$. For example, we can take

$$f_1(m) = \begin{cases} \sum\limits_{\ell=0}^{m-1} \frac{\ell}{c(T^\ell\omega)}, & if \quad m \geq 1 \\ 0, & if \quad m = 0 \\ \sum\limits_{\ell=1}^{-m} \frac{\ell}{c(T^{-\ell}\omega)}, & if \quad m \leq -1 \end{cases}$$

It is easy to check that $L_\omega f_1(m) = 1$ for any $m \in \mathbb{Z}$.

Lemma 5.2.1. *With function f_1 defined as above, we have .*

$$\mathbb{E}_\omega\left\{f_1(X_t)\right\} = t \tag{5.27}$$

for any $t \geq 0$.

Proof. Put $h_1(t) = \mathbb{E}_\omega \{f_1(X_t)\}$ then

$$\begin{aligned} L_\omega f_1(X_t) &= \lim_{s\to 0} \mathbb{E}_\omega \left\{ \frac{\mathbb{E}_\omega(f_1(X_{t+s})/X_t) - f_1(X_t)}{s} \right\} \\ &= \lim_{s\to 0} \mathbb{E}_\omega \left\{ \frac{f_1(X_{t+s}) - f_1(X_t)}{s} \right\} \\ &= \lim_{s\to 0} \frac{h_1(t+s) - h_1(t)}{s} = h_1'(t). \end{aligned}$$

Since $L_\omega f_1(X_t) = 1$ then $h_1'(t) = 1, \forall t$ implies $h_1(t) = t+c, \forall t$. Since $h_1(0) = \mathbb{E}_\omega \{f_1(X_0)\} = 0$ implies $c = 0$, and hence $h_1(t) = \mathbb{E}_\omega \{f_1(X_t)\} = t$. ☐

The formula (5.27) can be rewritten as

$$\mathbb{E}_\omega \left\{ \frac{f_1(X_t)}{X_t^2} \times \frac{X_t^2}{t} \right\} = 1$$

and note that if $\lim_{m\to\infty} \frac{f_1(m)}{m^2}$ exists then so $\lim_{n\to+\infty} \mathbb{E}_\omega \left\{ \frac{X_t^2}{t} \right\}$.

The next step we will compute the limit of $\frac{f_1(m)}{m^2}$ by using the pointwise ergodic theorem and lemma 5.1.1.

Lemma 5.2.2. *With function f_1 defined as above*

$$\lim_{m\to\pm\infty} \frac{f_1(m)}{m^2} = \sigma^{-2}. \tag{5.28}$$

Proof. Consider firstly the case $m > 0$. Applying lemma 5.1.1 for $u_\ell = \frac{1}{c(T^\ell\omega)}$, $v_\ell = 1$ and $\alpha = 1$, one has

$$\lim_{m\to+\infty} \frac{f_1(m)}{m^2} = \lim_{m\to+\infty} \frac{1}{m^2} \sum_{\ell=0}^{m-1} \frac{\ell}{c(T^\ell\omega)} = \frac{1}{2} \int_\Omega \frac{1}{c}\, d\mu.$$

Similarly, one has the same result for the case $m < 0$. ☐

From lemma 5.2.2, for any $\varepsilon > 0$, there exists $M > 0$ such that for any $m > M$ then

$$\left| \frac{m^2}{f_1(m)} - \sigma^2 \right| < \varepsilon/2. \tag{5.29}$$

Now we combine (5.27) and (5.29) to prove theorem 5.1.2. Put

$$H_1 = \mathbb{E}_\omega \left\{ \frac{X_t^2}{t} \mathbb{1}_{\{|X_t|>M\}} \right\} - \mathbb{E}_\omega \left\{ \sigma^2 \frac{f_1(X_t)}{t} \mathbb{1}_{\{|X_t|>M\}} \right\}$$

$$H_2 = \mathbb{E}_\omega \left\{ \frac{X_t^2}{t} \mathbb{1}_{\{|X_t|\le M\}} \right\} - \mathbb{E}_\omega \left\{ \sigma^2 \frac{f_1(X_t)}{t} \mathbb{1}_{\{|X_t|\le M\}} \right\}.$$

For t large enough

$$|H_1| = \left| \mathbb{E}_\omega \left\{ \left(\frac{X_t^2}{f_1(X_t)} - \sigma^2 \right) \frac{f_1(X_t)}{t} \mathbb{1}_{\{|X_t|>M\}} \right\} \right|$$

$$\begin{aligned} &\leq \mathbb{E}_\omega \left\{ \left| \frac{X_t^2}{f_1(X_t)} - \sigma^2 \right| \frac{f_1(X_t)}{t} \mathbb{1}_{\{|X_t|>M\}} \right\} \\ &< \varepsilon/2 \end{aligned}$$

since $f_1 \geq 0$, and

$$|H_2| = \left| \frac{1}{t} \mathbb{E}_\omega \left\{ \left(X_t^2 - \sigma^2 f_1(X_t) \right) \mathbb{1}_{\{|X_t| \leq M\}} \right\} \right| < \varepsilon/2$$

It follows that

$$\left| \mathbb{E}_\omega \left\{ \frac{X_t^2}{t} \right\} - \sigma^2 \right| = |H_1 + H_2| \leq |H_1| + |H_2| < \varepsilon$$

for t large enough. Since ε is as small as we need, then $\lim\limits_{t \to +\infty} \mathbb{E}_\omega \left\{ \frac{X_t^2}{t} \right\} = \sigma^2$. □

Theorem 5.2.3. *For almost all environments ω, we have*

$$\lim_{n \to +\infty} \mathbb{E}_\omega \left\{ \left(\frac{X_t^2}{t} \right)^k \right\} = \frac{(2k)!}{k! 2^k} \sigma^{2k} \tag{5.30}$$

for any $k \geq 1$.

Proof. Fix $\omega \in \Omega$.

We consider a sequence of functions $f_k \geq 0$, defined on $\mathbb{Z}$, such that $L_\omega f_k \equiv f_{k-1}$, $f_k(0) = 0$ and f_1 is defined as above. For example, we can take

$$\begin{aligned} f_0(m) &= 1 \quad \forall m \in \mathbb{Z} \\ f_1(m) &= \begin{cases} \sum\limits_{\ell=0}^{m-1} \frac{\ell}{c(T^\ell \omega)} f_0(s), & if \quad m \geq 1 \\ 0, & if \quad m = 0 \\ \sum\limits_{\ell=1}^{-m} \frac{\ell}{c(T^{-\ell} \omega)} f_0(-s) & if \quad m \leq -1 \end{cases} \\ &\dots \\ f_k(m) &= \begin{cases} \sum\limits_{\ell=0}^{m-1} \frac{1}{c(T^\ell \omega)} \sum\limits_{s=1}^{\ell} f_{k-1}(s), & if \quad m \geq 1 \\ 0, & if \quad m = 0 \\ \sum\limits_{\ell=1}^{-m} \frac{1}{c(T^{-\ell} \omega)} \sum\limits_{s=1}^{\ell-1} f_{k-1}(-s), & if \quad m \leq -1 \end{cases} \end{aligned}$$

Then it is easy to check that $L_\omega f_k(m) = f_{k-1}(m)$ for any $m \in \mathbb{Z}$.

Lemma 5.2.3. *For each $k \geq 1$, then*

$$\mathbb{E}_\omega \{ f_k(X_t) \} = \frac{t^k}{k!} \tag{5.31}$$

for any $t \geq 0$.

Proof. It is obvious to work with $k = 1$. Assume that it is true with $k \geq 1$, we claim that it is also with $k + 1$. That means: if

$$\mathbb{E}_\omega \left\{ f_k(X_t) \right\} = \frac{t^k}{k!}$$

then

$$\mathbb{E}_\omega \left\{ f_{k+1}(X_t) \right\} = \frac{t^{k+1}}{(k+1)!}.$$

Put $h_k(t) = \mathbb{E}_\omega \left\{ f_k(X_t) \right\}$ for $k \geq 1$ then

$$\begin{aligned} L_\omega f_{k+1}(X_t) &= \lim_{s \to 0} \mathbb{E}_\omega \left\{ \frac{\mathbb{E}_\omega (f_{k+1}(X_{t+s})/X_t) - f_{k+1}(X_t)}{s} \right\} \\ &= \lim_{s \to 0} \mathbb{E}_\omega \left\{ \frac{f_{k+1}(X_{t+s}) - f_{k+1}(X_t)}{s} \right\} \\ &= \lim_{s \to 0} \frac{h_{k+1}(t+s) - h_{k+1}(t)}{s} \\ &= h'_{k+1}(t). \end{aligned}$$

Since $L_\omega f_k(X_t) = \frac{t^k}{k!}$ then $h'_{k+1}(t) = \frac{t^k}{k!}, \forall t$ implies $h_{k+1}(t) = \frac{t^{k+1}}{(k+1)!} + c$. Since $h_{k+1}(0) = \mathbb{E}_\omega \left\{ f_{k+1}(X_0) \right\} = 0$ implies $c = 0$ and hence $h_{k+1}(t) = \frac{t^{k+1}}{(k+1)!}$. □

The formula (5.31) can be rewritten as

$$\mathbb{E}_\omega \left\{ \frac{f_k(X_t)}{X_t^{2k}} \times \frac{X_t^{2k}}{t^k} \right\} = \frac{1}{k!}$$

and note that if $\lim_{m \to \infty} \frac{f_k(m)}{m^{2k}}$ exists then so $\lim_{n \to +\infty} \mathbb{E}_\omega \left\{ \left(\frac{X_t^2}{t} \right)^k \right\}$.

The next step we will compute the limit of $\frac{f_k(m)}{m^{2k}}$ by using the pointwise ergodic theorem and lemma 5.1.1.

Lemma 5.2.4. *For each $k \geq 1$,*

$$\lim_{m \to \pm\infty} \frac{f_k(m)}{m^{2k}} = \frac{2^k}{(2k)!} \sigma^{-2k} = F_k \tag{5.32}$$

Proof. This limit is true for $k = 1$ (lemma 5.2.2).

Assume that (5.32) is also true for $k \geq 1$, we claim that it holds for $k + 1$, that is

$$\lim_{m \to \pm\infty} \frac{f_{k+1}(m)}{m^{2(k+1)}} = \frac{2^{k+1}}{(2(k+1))!} \sigma^{-2(k+1)}. \tag{5.33}$$

Consider firstly the case $m > 0$. Applying lemma 5.1.1 for $u_s = 1$, $v_s = \frac{1}{s^{2k}} f_k(s)$ and $\alpha = 2k$, one has

$$\lim_{\ell \to +\infty} \frac{1}{\ell^{2k+1}} \sum_{s=1}^{\ell} f_k(s) = \lim_{\ell \to +\infty} \frac{1}{\ell^{2k+1}} \sum_{s=1}^{\ell} s^{2k} \frac{1}{s^{2k}} f_k(s) = \frac{2^k}{(2k+1)!} \sigma^{-2k}.$$

Again, applying lemma 5.1.1 for $u'_\ell = \frac{1}{c(T^\ell\omega)}$, $v'_\ell = \frac{1}{\ell^{2k+1}}\sum_{s=1}^{\ell} f_k(s)$ and $\alpha = 2k+1$, one has

$$\begin{aligned}\lim_{m\to+\infty}\frac{f_{2k+1}(m)}{m^{2(k+1)}} &= \lim_{m\to+\infty}\frac{1}{m^{2(k+1)}}\sum_{\ell=0}^{m-1}\frac{1}{c(T^\ell\omega)}\sum_{s=1}^{\ell} f_k(s)\\ &= \lim_{m\to+\infty}\frac{1}{m^{2(k+1)}}\sum_{\ell=0}^{m-1}\frac{\ell^{2k+1}}{c(T^\ell\omega)}\frac{1}{\ell^{2k+1}}\sum_{s=1}^{\ell} f_k(s)\\ &= \frac{2^{k+1}}{(2(k+1))!}\sigma^{-2(k+1)}.\end{aligned}$$

Similarly, one has the same result for the case $m < 0$. □

From lemma 5.2.4, for any $\varepsilon > 0$, there exists $M > 0$ such that for any $|m| > M$ then

$$\left|\frac{m^{2k}}{f_k(m)} - \frac{1}{F_k}\right| < \varepsilon/2. \tag{5.34}$$

Now we conbine (5.31) and (5.34) to prove theorem 5.2.3. Put

$$H_3 = \mathbb{E}_\omega\left\{\frac{X_t^{2k}}{t^k}\mathbb{1}_{\{|X_t|>M\}}\right\} - \mathbb{E}_\omega\left\{\frac{f_k(X_t)}{t^kF_k}\mathbb{1}_{\{|X_t|>M\}}\right\}$$

$$H_4 = \mathbb{E}_\omega\left\{\frac{X_t^{2k}}{t^k}\mathbb{1}_{\{|X_t|\le M\}}\right\} - \mathbb{E}_\omega\left\{\frac{f_k(X_t)}{t^kF_k}\mathbb{1}_{\{|X_t|\le M\}}\right\}.$$

For t large enough

$$\begin{aligned}|H_3| &= \left|\mathbb{E}_\omega\left\{\left(\frac{X_t^{2k}}{f_k(X_t)} - \frac{1}{F_k}\right)\frac{f_k(X_t)}{t^k}\mathbb{1}_{\{|X_t|>M\}}\right\}\right|\\ &\le \mathbb{E}_\omega\left\{\left|\frac{X_t^{2k}}{f_k(X_t)} - \frac{1}{F_k}\right|\frac{f_k(X_t)}{t^k}\mathbb{1}_{\{|X_t|>M\}}\right\}\\ &< \varepsilon/2\end{aligned}$$

since $f_k \ge 0$, and

$$|H_4| = \frac{1}{t^k}\left|\mathbb{E}_\omega\left\{\left(X_t^{2k} - \frac{1}{F_k}f_k(X_t)\right)\mathbb{1}_{\{|X_t|\le M\}}\right\}\right| < \varepsilon/2.$$

It follows that

$$\left|\mathbb{E}_\omega\left\{\left(\frac{X_t^2}{t}\right)^k\right\} - \frac{1}{k!F_k}\right| = |H_3 + H_4| \le |H_3| + |H_4| < \varepsilon$$

for t large enough. Since ε is as small as we need, then $\lim_{t\to+\infty}\mathbb{E}_\omega\left\{\left(\frac{X_t^2}{t}\right)^k\right\} = \frac{(2k)!}{k!2^k}\sigma^{2k}$. □

Theorem 5.2.4. *For almost all environments ω, we have*

$$\lim_{t\to+\infty}\mathbb{E}_\omega\left\{\frac{X_t}{\sqrt{t}}\right\} = 0 \tag{5.35}$$

Proof. Fix $\omega \in \Omega$.

We consider a function g_1, defined on $\mathbb{Z}$, satisfying $L_\omega g_1 \equiv 0$ and $g_1(0) = 0$. For instance, we can take

$$g_1(m) = \begin{cases} \sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell \omega)}, & if \quad m \geq 1 \\ 0, & if \quad m = 0 \\ -\sum_{\ell=1}^{-m} \frac{1}{c(T^{-\ell}\omega)}, & if \quad m \leq -1 \end{cases}$$

Put $q_1(t) = \mathbb{E}_\omega \{g_1(X_t)\}$, then

$$\begin{aligned} L_\omega g_1(X_t) &= \lim_{s\to 0} \mathbb{E}_\omega \left\{ \frac{\mathbb{E}_\omega(g_1(X_{t+s})/X_t) - g_0(X_t)}{s} \right\} \\ &= \lim_{s\to 0} \mathbb{E}_\omega \left\{ \frac{g_1(X_{t+s}) - g_1(X_t)}{s} \right\} \\ &= \lim_{s\to 0} \frac{q_1(t+s) - q_0(t)}{s} = q_1'(t) \end{aligned}$$

Since $L_\omega g_1(X_t) = 0$ then $q_1'(t) = 0, \forall t$ implies $q_1(t) = c, \forall t$. Since $q_1(0) = \mathbb{E}_\omega \{g_1(X_0)\} = 0$ implies $c = 0$, and hence

$$\mathbb{E}_\omega \{g_1(X_t)\} = q_1(t) = 0. \tag{5.36}$$

The formula (5.36) can be rewritten as

$$\mathbb{E}_\omega \left\{ \frac{g_1(X_t)}{X_t} \times \frac{X_t}{\sqrt{t}} \right\} = 0$$

and note that if $\lim_{m\to\infty} \frac{g_1(m)}{m}$ exists then $\lim_{n\to+\infty} \mathbb{E}_\omega \left\{ \frac{X_t}{\sqrt{t}} \right\}$ exists.

The pointwise ergodic theorem ensures that

$$\lim_{m\to\infty} \frac{g_1(m)}{m} = \int_\Omega \frac{1}{c}\, d\mu = G_1$$

Therefore, for any $\varepsilon > 0$, there exists $M > 0$ such that for any $|m| > M$ then

$$\left| \frac{g_1(m)}{G_1 m} - 1 \right| < \varepsilon. \tag{5.37}$$

We now combine (5.36) and (5.37) to prove theorem 5.2.4. Put

$$H_5 = \mathbb{E}_\omega \left\{ \frac{1}{\sqrt{t}} X_t \mathbb{1}_{\{|X_t| \leq M\}} \right\} - \mathbb{E}_\omega \left\{ \frac{1}{\sqrt{t}} \frac{g_1(X_t)}{G_1} \mathbb{1}_{\{|X_t| \leq M\}} \right\}$$

and

$$H_6 = \mathbb{E}_\omega \left\{ \frac{1}{\sqrt{t}} X_t \mathbb{1}_{\{|X_t| > M\}} \right\} - \mathbb{E}_\omega \left\{ \frac{1}{\sqrt{t}} \frac{g_1(X_t)}{G_1} \mathbb{1}_{\{|X_t| > M\}} \right\}.$$

We have

$$|H_5| = \left| \mathbb{E}_\omega \left\{ \frac{1}{\sqrt{t}} \left(X_t - \frac{g_1(X_t)}{G_1} \right) \mathbb{1}_{\{|X_t| \leq M\}} \right\} \right| < \varepsilon$$

and

$$\begin{aligned} |H_6| &= \left|\mathbb{E}_\omega \left\{ \frac{1}{\sqrt{t}} \left(1 - \frac{g_1(X_t)}{X_t G_1} \right) X_t \mathbb{1}_{\{|X_t|>M\}} \right\}\right| \\ &\le \varepsilon \mathbb{E}_\omega \left\{ \frac{|X_t|}{\sqrt{t}} \right\} \le \varepsilon \sqrt{\mathbb{E}_\omega \left\{ \frac{X_t^2}{t} \right\}} \end{aligned}$$

for t large enough. It follows that

$$\left|\mathbb{E}_\omega \left\{ \frac{X_t}{\sqrt{t}} \right\}\right| = |H_5 + H_6| \le |H_5| + |H_6| < \varepsilon + \varepsilon \sqrt{\mathbb{E}_\omega \left\{ \frac{X_t^2}{t} \right\}}$$

for t large enough. By theorem 5.2.2 $\lim\limits_{t\to+\infty} \mathbb{E}_\omega \left\{ \frac{X_t^2}{t} \right\} = \sigma^2 < \infty$ and ε is as small as we need, then we obtain $\lim\limits_{t\to+\infty} \mathbb{E}_\omega \left\{ \frac{X_t}{\sqrt{t}} \right\} = 0$. □

Theorem 5.2.5. *For almost all environments ω, we have*

$$\lim_{t\to+\infty} \mathbb{E}_\omega \left\{ \left(\frac{X_t}{\sqrt{t}} \right)^{(2k-1)} \right\} = 0 \tag{5.38}$$

for each $k \ge 1$.

Proof. Fix $\omega \in \Omega$. We consider a sequence of functions g_k, defined on $\mathbb{Z}$, satisfying $L_\omega g_{k+1} \equiv g_k$, $\forall k \ge 1$ and g_1 is defined as above. For instance, we can take

$$g_{k+1}(m) = \begin{cases} \sum\limits_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)} \sum\limits_{s=1}^{\ell} g_k(s), & if \quad m \ge 1 \\ 0, & if \quad m = 0 \\ \sum\limits_{\ell=1}^{-m} \frac{1}{c(T^{-\ell}\omega)} \sum\limits_{s=1}^{\ell-1} g_k(-s), & if \quad m \le -1 \end{cases}$$

Lemma 5.2.5. *For each $k \ge 1$, then*

$$\mathbb{E}_\omega \{g_k(X_t)\} = 0 \tag{5.39}$$

for any $t \ge 0$.

Proof. It is obvious to work with $k = 1$. Assume that it is true with $k \ge 1$, we claim that it is also with $k + 1$. That means: if

$$\mathbb{E}_\omega \{g_k(X_t)\} = 0$$

then

$$\mathbb{E}_\omega \{g_{k+1}(X_t)\} = 0.$$

Similarly, put $q_k(t) = \mathbb{E}_\omega \{g_k(X_t)\}$ for $k \ge 1$then

$$L_\omega g_{k+1}(X_t) = \lim_{s\to 0} \mathbb{E}_\omega \left\{ \frac{\mathbb{E}_\omega(g_{k+1}(X_{t+s})/X_t) - g_{k+1}(X_t)}{s} \right\}$$

$$
\begin{aligned}
&= \lim_{s\to 0} \mathbb{E}_\omega \left\{ \frac{g_{k+1}(X_{t+s}) - g_{k+1}(X_t)}{s} \right\} \\
&= \lim_{s\to 0} \frac{q_{k+1}(t+s) - q_{k+1}(t)}{s} = q'_{k+1}(t).
\end{aligned}
$$

Since $L_\omega g_k(X_t) = 0$ then $q'_{k+1}(t) = 0$, for any t implies $q_{k+1}(t) = c$ (constant). Since $q_k(0) = E_\omega \{g_k(X_0)\} = 0$ implies $c = 0$, and hence $q_k(t) = \mathbb{E}_\omega \{g_k(X_t)\} = 0$, for any t. □

The formula (5.39) can be rewritten as

$$
\mathbb{E}_\omega \left\{ \frac{g_k(X_t)}{X_t^{2k-1}} \times \frac{X_t^{2k-1}}{\left(\sqrt{t}\right)^{2k-1}} \right\} = \frac{1}{k!}
$$

and note that if $\lim_{m\to\infty} \frac{g_k(m)}{m^{2k-1}}$ exists then so $\lim_{n\to+\infty} \mathbb{E}_\omega \left\{ \left(\frac{X_t}{\sqrt{t}}\right)^{2k-1} \right\}$.

The next step we will compute the limit of $\frac{f_k(m)}{m^{2k}}$ by using the pointwise ergodic theorem and lemma 5.1.1.

Lemma 5.2.6. *For each $k \geq 1$ one has*

$$
\lim_{m\to\infty} \frac{g_k(m)}{m^{2k-1}} = \frac{1}{(2k-1)!} \left[\int_\Omega \frac{1}{c}\, d\mu \right]^k = G_k \tag{5.40}
$$

Proof. This limit is true for $k = 1$.

Assume that (5.40) is also true for $k \geq 1$, we claim that it holds for $k+1$, that is

$$
\lim_{m\to+\infty} \frac{g_{k+1}(m)}{m^{2k+1}} = \frac{1}{(2k+1)!} \left[\int_\Omega \frac{1}{c}\, d\mu \right]^{k+1}. \tag{5.41}
$$

Consider firstly the case $m > 0$. Applying lemma 5.1.1 for $u_s = 1$, $v_s = \frac{1}{s^{2k-1}} g_k(s)$ and $\alpha = 2k-1$, one has

$$
\begin{aligned}
\lim_{\ell\to+\infty} \frac{1}{\ell^{2k}} \sum_{s=1}^{\ell} g_k(s) &= \lim_{\ell\to+\infty} \frac{1}{\ell^{2k}} \sum_{s=1}^{\ell} s^{2k-1} \frac{1}{s^{2k-1}} g_k(s) \\
&= \frac{1}{(2k)!} \left[\int_\Omega \frac{1}{c}\, d\mu \right]^k.
\end{aligned}
$$

Again, applying lemma 5.1.1 for $u'_\ell = \frac{1}{c(T^\ell\omega)}$, $v'_\ell = \frac{1}{\ell^{2k}} \sum_{s=1}^{\ell} g_k(s)$ and $\alpha = 2k$, one has

$$
\begin{aligned}
\lim_{m\to+\infty} \frac{g_{k+1}(m)}{m^{2k+1}} &= \lim_{m\to+\infty} \frac{1}{m^{2k+1}} \sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)} \sum_{s=1}^{\ell} g_k(s) \\
&= \lim_{m\to+\infty} \frac{1}{m^{2k+1}} \sum_{\ell=0}^{m-1} \frac{\ell^{2k}}{c(T^\ell\omega)} \frac{1}{\ell^{2k}} \sum_{s=1}^{\ell} g_k(s) \\
&= \frac{1}{(2k+1)!} \left[\int_\Omega \frac{1}{c}\, d\mu \right]^{k+1}.
\end{aligned}
$$

Similarly, one has the same result for the case $m < 0$. □

From lemma 5.2.6, for any $\varepsilon > 0$, there exists $M > 0$ such that for any $|m| > M$ then

$$\left| \frac{g_k(m)}{m^{2k-1} G_k} - 1 \right| < \varepsilon. \tag{5.42}$$

We now combine (5.39) and (5.42) to prove theorem 5.2.5. Put

$$H_7 = \mathbb{E}_\omega \left\{ \frac{1}{(\sqrt{t})^{2k-1}} X_t^{2k-1} \mathbb{1}_{\{|X_t| \le M\}} \right\} - \mathbb{E}_\omega \left\{ \frac{1}{(\sqrt{t})^{2k-1}} \frac{g_k(X_t)}{G_k} \mathbb{1}_{\{|X_t| \le M\}} \right\}$$

and

$$H_8 = \mathbb{E}_\omega \left\{ \frac{1}{(\sqrt{t})^{2k-1}} X_t^{2k-1} \mathbb{1}_{\{|X_t| > M\}} \right\} - \mathbb{E}_\omega \left\{ \frac{1}{(\sqrt{t})^{2k-1}} \frac{g_k(X_t)}{G_k} \mathbb{1}_{\{|X_t| > M\}} \right\}.$$

We have

$$|H_7| = \left| \mathbb{E}_\omega \left\{ \frac{1}{(\sqrt{t})^{2k+1}} \left(X_t^{2k-1} - \frac{g_k(X_t)}{G_k} \right) \mathbb{1}_{\{|X_t| \le M\}} \right\} \right| < \varepsilon$$

and

$$\begin{aligned} |H_8| &= \left| \mathbb{E}_\omega \left\{ \frac{1}{(\sqrt{t})^{2k-1}} \left(1 - \frac{g_k(X_t)}{X_t^{2k-1} G_k} \right) X_t^{2k-1} \mathbb{1}_{\{|X_t| > M\}} \right\} \right| \\ &< \varepsilon \mathbb{E}_\omega \left\{ \left| \frac{X_t}{\sqrt{t}} \right|^{2k-1} \right\} \le \varepsilon \sqrt{ \mathbb{E}_\omega \left\{ \left(\frac{X_t^2}{t} \right)^{2k-1} \right\} } \end{aligned}$$

for t large enough. It follows that

$$\left| \mathbb{E}_\omega \left\{ \left(\frac{X_t}{\sqrt{t}} \right)^{(2k-1)} \right\} \right| = |H_7 + H_8| \le |H_7| + |H_8| < \varepsilon + \varepsilon \sqrt{ \mathbb{E}_\omega \left\{ \left(\frac{X_t^2}{t} \right)^{2k-1} \right\} }$$

for t large enough. By theorem 5.2.3 one has

$$\lim_{t \to +\infty} \mathbb{E}_\omega \left\{ \left(\frac{X_t^2}{t} \right)^{2k-1} \right\} = \frac{(2(2k-1))!}{(2k-1)! 2^{(2k-1)}} \sigma^{2(2k-1)}$$

and ε is as small as we need, then $\lim_{t \to +\infty} \mathbb{E}_\omega \left\{ \left(\frac{X_t}{\sqrt{t}} \right)^{(2k-1)} \right\} = 0.$ □

Finally, for each $\ell = 1, 2, 3, \dots$ we obtain

$$\lim_{t \to +\infty} \mathbb{E}_\omega \left\{ \left(\frac{X_t}{\sqrt{t}} \right)^{\ell} \right\} = \begin{cases} 0 & if \quad \ell = 2k-1 \\ \frac{(2k)!}{2^k k!} \sigma^{2k} & if \quad \ell = 2k \end{cases}$$

And hence, for almost all environment ω

$$\frac{X_t}{\sqrt{t}} \xrightarrow{\quad D \quad} \mathcal{N}\left(0, \sigma^2\right) \quad as\ t \longrightarrow +\infty$$

which completes the proof of theorem 5.2.1.

□

Chapter 6

Einstein's relation for reversible diffusions in a random environment in one dimension

This chapter is devoted to consider reversible diffusions in a random environment in one dimension and prove the Einstein's relation for this model. It says that the derivative at 0 of the effective velocity under an additional local drift equals the diffusivity of the model without drift (theorem 6.1.1). This equality was used by Einstein to measure the Avogadro number. Our method here is to solve the Poisson's equation $(P_\omega - I)g = f$ which introduced in the preceding chapter, and then use the pointwise ergodic theorem to treat the limit of the solutions to obtain the desired result.

6.1 Introduction

Consider again, on the $\mathbb{Z}$ network, a random stationary sequence of conductances, defined on a probability space $(\Omega, \mathcal{A}, \mu)$, an invertible $\mu-$preserving transformation T which is also ergodic, and a positive measurable function c on Ω. The space Ω is interpreted as the space of environments.

For a fixed environment $\omega \in \Omega$ and a fixed number $\lambda \neq 0$, the conductances of the edges $[k, k+1]$ is $e^{\lambda} c(T^k \omega)$ and $[k, k-1]$ is $e^{-\lambda} c(T^{k-1}\omega)$. The number λ is called the "drift" of the model.

We consider Markov process $(X_t)_{t\geq 0}$ on $\mathbb{Z}$ with $X_0 = 0$, the generator infinitesimal

$$L_{\lambda,\omega} f(k) = e^{-\lambda} c(T^{k-1}\omega) f(k-1) + e^{\lambda} c(T^k \omega) f(k+1) - \pi(T^k\omega) f(k), \tag{6.1}$$

where $\pi = e^{\lambda} c + e^{-\lambda} c \circ T^{-1}$.

Definition 6.1.1. *The Quenched diffusivity of a diffusion process X_t without drift is defined by*

$$\Sigma = \lim_{t\to+\infty} \frac{1}{t} \mathbb{E}_\omega \left\{ X_t^2 \right\} \tag{6.2}$$

***Remark* 6.1.1.** *When the model without drift $\lambda = 0$, in the preceding chapter theorems 5.1.2 and 5.2.2 show that for almost all environment ω*

$$\Sigma = \lim_{t \to +\infty} \frac{1}{t} \mathbb{E}_\omega \left\{ X_t^2 \right\} = \sigma^2 \tag{6.3}$$

where

$$\sigma^2 = \left[\begin{array}{l} \left[\int \frac{1}{c}\, d\mu \int c\, d\mu\right]^{-1} \text{ if } c,\ c^{-1} \in L^1(\mu) \text{ and } X_n \text{ is a random walk.} \\ \\ 2\left[\int \frac{1}{c}\, d\mu\right]^{-1} \text{ if } c^{-1} \in L^1(\mu) \text{ and } X_t \text{ is a Markov process with time continuous.} \end{array} \right.$$

Definition 6.1.2. *The Quenched effective drift of a diffusion process X_t in $\mathbb{Z}$ is defined by*

$$d_\omega(\lambda) = \lim_{t \to +\infty} \frac{1}{t} \mathbb{E}_{\lambda,\omega} \left\{ X_t \right\}. \tag{6.4}$$

***Remark* 6.1.2.** *When the model without drift $\lambda = 0$, then $d_\omega(0) = \lim_{t\to+\infty} \frac{1}{t}\mathbb{E}_\omega \left\{X_t\right\} = 0$ with the same condition of function c in remark 6.1.1. It was defined in the preceding chapter (theorems 5.1.4 and 5.2.4).*

Theorem 6.1.1. ***(Einstein's relation)*** *The function $\lambda \longmapsto d_\omega(\lambda)$ has a derivative at $\lambda = 0$ which satisfies*

$$\lim_{\lambda \to 0} \frac{d_\omega(\lambda)}{\lambda} = \Sigma = \sigma^2 \tag{6.5}$$

if $c,\ c^{-1} \in L^2(\mu)$ for a random walk and $c^{-1} \in L^1(\mu)$ for a Markov process with time continuous respectively.

This theorem will be proved into two cases:

- For Random walk in Random environment with a drift, we have theorem 6.2.1.
- For Markov processes in Random environment with a drift, we have theorem 6.3.1.

We will see in the proof of these theorems that $d_\omega(\lambda)$ is defined *a.s* and doesn't depend on ω. So, it will be denoted by $d(\lambda)$ in the sequel.

***Remark* 6.1.3.** *About Einstein's relation for reversible diffusions in random environment, there is a paper of Gantert, Mathieu, Piatnitski [19] recently. They used independence's assumption in the environment.*

6.2 Random walk in Random environment with a drift

We introduce the random walk $(X_n)_{n\geq 0}$ on $\mathbb{Z}$ with initial condition $X_0 = 0$ and Markov's operator $f \longmapsto P_{\lambda,\omega} f$ defined by

$$P_{\lambda,\omega} f(k) = \frac{1}{\pi(T^k\omega)} \left[e^{-\lambda} c(T^{k-1}\omega) f(k-1) + e^{\lambda} c(T^k\omega) f(k+1) \right]. \tag{6.6}$$

Theorem 6.2.1. *For almost all environment ω,*

$$\lim_{\lambda\to 0}\frac{d(\lambda)}{\lambda}=\lim_{\lambda\to 0}\frac{1}{\lambda}\lim_{n\to\infty}\mathbb{E}_{\lambda,\omega}\left\{\frac{X_n}{n}\right\}=\left[\int_\Omega c\,d\mu\int_\Omega\frac{1}{c}\,d\mu\right]^{-1}. \tag{6.7}$$

if $c,\ c^{-1}\in L^2(\mu)$.

Proof. This theorem is proved by Theorems 6.2.2 and 6.2.3. □

Theorem 6.2.2. *For almost all environment ω and for $\lambda>0$*

$$\lim_{\lambda\to 0^+}\frac{d(\lambda)}{\lambda}=\lim_{\lambda\to 0^+}\frac{1}{\lambda}\lim_{n\to\infty}\mathbb{E}_{\lambda,\omega}\left\{\frac{X_n}{n}\right\}=\left[\int_\Omega c\,d\mu\int_\Omega\frac{1}{c}\,d\mu\right]^{-1}. \tag{6.8}$$

Proof. Fix $\omega\in\Omega$. We consider a functions f_λ, defined on $\mathbb{Z}$, such that $(P_{\lambda,\omega}-I)f_\lambda\equiv 1$ and $f_\lambda(0)=0$. For example, we can take

$$f_\lambda(m)=\begin{cases}\sum\limits_{\ell=0}^{m-1}\frac{1}{c(T^\ell\omega)e^{2\ell\lambda}}\sum\limits_{s=-\infty}^{\ell}\pi(T^s\omega)e^{(2s-1)\lambda}, & if \quad m\geq 1\\ 0, & if \quad m=0\\ -\sum\limits_{\ell=1}^{-m}\frac{1}{c(T^{-\ell}\omega)}e^{2\ell\lambda}\sum\limits_{s=-\infty}^{-\ell}\pi(T^s\omega)e^{(2s-1)\lambda}, & if \quad m\leq -1\end{cases}$$

It is easy to check that $(P_{\lambda,\omega}-I)f_\lambda(m)=1$ for any $m\in\mathbb{Z}$. Replacing m by X_n and take the expectation, one has

$$\mathbb{E}_{\lambda,\omega}\left\{f_\lambda(X_n)\right\}=n\quad\forall n\geq 0. \tag{6.9}$$

The formula (6.9) can be rewritten as

$$\mathbb{E}_\omega\left\{\frac{f_\lambda(X_n)}{X_n}\times\frac{X_n}{n}\right\}=1$$

and note that if $\lim_{m\to\infty}\frac{f_\lambda(m)}{m}$ exists then so $\lim_{n\to+\infty}\mathbb{E}_\omega\left\{\frac{X_n}{n}\right\}$.

The next step we will compute the limit of $\frac{f_\lambda(m)}{m}$ by using the pointwise ergodic theorem. We need the following lemma in the proof:

Lemma 6.2.1. *Let $(a_n)_{n\geq 0}$ be a sequence of positive real numbers and let A_n be a partial sum $A_n=\sum\limits_{i=0}^{n}a_i$. Assume that $\lim\limits_{n\to\infty}\frac{1}{n}A_n=L$ then*

$$\sum_{\ell=0}^{+\infty}a_\ell\rho^\ell<+\infty \tag{6.10}$$

and

$$\sum_{\ell=0}^{+\infty}a_\ell\ell\rho^\ell<+\infty \tag{6.11}$$

where $0<\rho<1$. Furthermore

$$\lim_{\rho\to 1^-}(1-\rho)\sum_{\ell=0}^{+\infty}a_\ell\rho^\ell=L. \tag{6.12}$$

Proof. It is clear that (6.10) is followed by (6.11).

Proof of (6.11). Applying Abel's lemma we have

$$\begin{aligned}\sum_{\ell=0}^{n} a_\ell \ell \rho^\ell &= \sum_{\ell=0}^{n-1} A_\ell \left(\ell\rho^\ell - (\ell+1)\rho^{\ell+1}\right) + A_n n \rho^n \\ &= (1-\rho)\sum_{\ell=0}^{n-1} A_\ell \ell \rho^\ell - \sum_{\ell=0}^{n-1} A_\ell \rho^{\ell+1} + A_n n \rho^n.\end{aligned}$$

Since $\lim_{n\to\infty} \frac{1}{n} A_n = L$ then $\lim_{n\to\infty} A_n n \rho^n = 0$, $\sum_{\ell=0}^{+\infty} A_\ell \ell \rho^\ell$ and $\sum_{\ell=0}^{+\infty} A_\ell \rho^{\ell+1}$ converge by D'Alembert criterion, which prove (6.11).

Furthermore, for any $\varepsilon > 0$ there exists $N > 0$ such that for any $n \geq N$ we have $\left|\frac{1}{n} A_n - L\right| < \varepsilon$ and $\lim_{n\to\infty} A_n \rho^n = 0$. Therefore

$$\begin{aligned}\left|(1-\rho)\sum_{\ell=0}^{+\infty} a_\ell \rho^\ell - L\right| &= \left|(1-\rho)^2 \sum_{\ell=0}^{\infty} A_\ell \rho^\ell - \frac{(1-\rho)^2}{\rho} L \sum_{\ell=0}^{+\infty} \ell \rho^\ell\right| \\ &= (1-\rho)^2 \left|\sum_{\ell=0}^{\infty} \left(\frac{A_\ell}{\ell} - \frac{1}{\rho} L\right) \ell \rho^\ell\right| \\ &\leq (1-\rho)^2 \sum_{\ell=0}^{N-1} \left|\frac{A_\ell}{\ell} - \frac{1}{\rho} L\right| \ell \rho^\ell + (1-\rho)^2 \sum_{\ell=N}^{\infty} \ell \rho^\ell \left(\frac{1}{\rho} L - L + \varepsilon\right) \\ &\leq (1-\rho)^2 \sum_{\ell=0}^{N-1} \left|\frac{A_\ell}{\ell} - \frac{1}{\rho} L\right| \ell \rho^\ell + (1-\rho) L + \varepsilon\end{aligned}$$

for $\rho \to 1^-$, (6.12) is followed.

□

In the sequel, we always assume that $\rho = \frac{1}{e^{2\lambda}}$ and by the pointwise ergodic theorem $\frac{1}{n}\sum_{k=0}^{n-1} \pi(T^{-k}\omega) = \int_\Omega \pi \, d\mu$. Therefore if we put $H_\lambda(\omega) = \sqrt{\rho}\sum_{k=0}^{+\infty} \pi(T^{-k}\omega)\rho^k$, lemma 6.2.1 shows that $H_\lambda(\omega) < +\infty$ and

$$\lim_{\lambda\to 0^+} (1 - e^{-2\lambda}) H_\lambda(\omega) = \lim_{\rho\to 1^-} (1-\rho) H_\lambda(\omega) = \int_\Omega \pi \, d\mu. \tag{6.13}$$

Lemma 6.2.2. *With function f_λ defined as above, we have*

$$\lim_{m\to\pm\infty} \frac{f_\lambda(m)}{m} = \int_\Omega \frac{H_\lambda}{c} d\mu = L_\lambda. \tag{6.14}$$

Proof. By the definition of function f_λ, for $m > 0$

$$\frac{f_\lambda(m)}{m} = \frac{1}{m}\sum_{\ell=0}^{m-1} \frac{\rho^\ell}{c(T^\ell\omega)} \sqrt{\rho} \sum_{s=-\infty}^{\ell} \pi(T^s\omega)\rho^{-s} = \frac{1}{m}\sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)} \sqrt{\rho} \sum_{s=-\infty}^{\ell} \pi(T^s\omega)\rho^{\ell-s}$$

$$= \frac{1}{m}\sum_{\ell=0}^{m-1}\frac{1}{c(T^\ell\omega)}\left[\sqrt{\rho}\sum_{s=-\infty}^{\ell}\pi(T^{s-\ell}\omega)\rho^{\ell-s}\right]\circ T^\ell.$$

Replacing $\ell - s$ by k we obtain

$$\frac{f_\lambda(m)}{m} = \frac{1}{m}\sum_{\ell=0}^{m-1}\frac{1}{c(T^\ell\omega)}\left[\sqrt{\rho}\sum_{k=0}^{+\infty}\pi(T^{-k}\omega)\rho^k\right]\circ T^\ell = \frac{1}{m}\sum_{\ell=0}^{m-1}\frac{H_\lambda}{c}\circ T^\ell(\omega)$$

By Cauchy-Schwarz's inequality, we have

$$H_\lambda^2(\omega) \leq \frac{\rho}{1-\rho}\sum_{k=0}^{+\infty}\pi^2(T^{-k}\omega)\rho^k$$

then

$$\int_\Omega H_\lambda^2 d\mu \leq \frac{\rho}{(1-\rho)^2}\int_\Omega \pi^2 d\mu.$$

Since $\pi \in L^2(\mu)$ then $H_\lambda \in L^2(\mu)$, and hence by Holder's inequality $\frac{H_\lambda}{c} \in L^1(\mu)$. It follows that

$$\lim_{m\to+\infty}\frac{f_\lambda(m)}{m} = \lim_{m\to+\infty}\frac{1}{m}\sum_{\ell=0}^{m-1}\frac{H_\rho}{c}\circ T^\ell(\omega) = \int_\Omega \frac{H_\lambda}{c}d\mu$$

by pointwise ergodic theorem. Similarly, for $m < 0$

$$\begin{aligned}
\frac{f_\lambda(m)}{m} &= \frac{1}{-m}\sum_{\ell=1}^{-m}\frac{\rho^{-\ell}}{c(T^{-\ell}\omega)}\sqrt{\rho}\sum_{s=-\infty}^{-\ell}\pi(T^s\omega)\rho^{-s}\\
&= \frac{1}{-m}\sum_{\ell=1}^{-m}\frac{1}{c(T^{-\ell}\omega)}\sqrt{\rho}\sum_{s=-\infty}^{-\ell}\pi(T^s\omega)\rho^{-s-\ell}\\
&= \frac{1}{-m}\sum_{\ell=1}^{-m}\frac{1}{c(T^{-\ell}\omega)}\left[\sqrt{\rho}\sum_{s=-\infty}^{-\ell}\pi(T^{s+\ell}\omega)\rho^{-s-\ell}\right]\circ T^{-\ell}.
\end{aligned}$$

Replacing $s + \ell$ by $-k$ we obtain

$$\frac{f_\lambda(m)}{m} = \frac{1}{-m}\sum_{\ell=1}^{-m}\frac{1}{c(T^{-\ell}\omega)}\left[\sqrt{\rho}\sum_{k=0}^{+\infty}\pi(T^{-k}\omega)\rho^k\right]\circ T^{-\ell} = \frac{1}{-m}\sum_{\ell=1}^{-m}\frac{H_\lambda}{c}\circ T^{-\ell}(\omega).$$

By pointwise ergodic theorem

$$\lim_{m\to-\infty}\frac{f_\lambda(m)}{m} = \lim_{m\to+\infty}\frac{1}{m}\sum_{\ell=1}^{m}\frac{H_\lambda}{c}\circ T^{-\ell}(\omega) = \int_\Omega\frac{H_\lambda}{c}d\mu = L_\lambda.$$

$\square$

For any $\varepsilon > 0$, by (6.14) there exists $M > 0$ such that for any $|m| > M$ then

$$\left| \frac{1}{L_\lambda} \frac{f_\lambda(m)}{m} - 1 \right| < \varepsilon. \tag{6.15}$$

We now combine (6.9) and (6.15) to compute the limit of $\mathbb{E}_{\lambda,\omega}\left\{\frac{X_n}{n}\right\}$. Put

$$I_1^\lambda = \mathbb{E}_{\lambda,\omega}\left\{\frac{X_n}{n}\mathbb{1}_{\{|X_n|\le M\}}\right\} - \mathbb{E}_{\lambda,\omega}\left\{\frac{1}{L_\lambda}\frac{f_\lambda(X_n)}{n}\mathbb{1}_{\{|X_n|\le M\}}\right\}$$

$$I_2^\lambda = \mathbb{E}_{\lambda,\omega}\left\{\frac{X_n}{n}\mathbb{1}_{\{|X_n|> M\}}\right\} - \mathbb{E}_{\lambda,\omega}\left\{\frac{1}{L_\lambda}\frac{f_\lambda(X_n)}{n}\mathbb{1}_{\{|X_n|> M\}}\right\}$$

then

$$\begin{aligned}
\left|I_1^\lambda\right| &= \left|\frac{1}{n}\mathbb{E}_{\lambda,\omega}\left\{\left[X_n - \frac{f_\lambda(X_n)}{L_\lambda}\right]\mathbb{1}_{\{|X_n|\le M\}}\right\}\right| \\
&\le \frac{1}{n}\mathbb{E}_{\lambda,\omega}\left\{\left|X_n - \frac{f_\lambda(X_n)}{L_\lambda}\right|\mathbb{1}_{\{|X_n|\le M\}}\right\} \\
&< \varepsilon
\end{aligned}$$

and

$$\begin{aligned}
\left|I_2^\lambda\right| &= \left|\mathbb{E}_{\lambda,\omega}\left\{\left(1 - \frac{1}{L_\lambda}\frac{f_\lambda(X_n)}{X_n}\right)\frac{X_n}{n}\mathbb{1}_{\{|X_n|> M\}}\right\}\right| \\
&< \varepsilon\sqrt{\mathbb{E}_{\lambda,\omega}\left\{\frac{X_n^2}{n^2}\right\}}
\end{aligned}$$

for n large enough. It follows that

$$\left|\mathbb{E}_{\lambda,\omega}\left\{\frac{X_n}{n}\right\} - \frac{1}{L_\lambda}\right| = \left|I_1^\lambda + I_2^\lambda\right| \le \left|I_1^\lambda\right| + \left|I_2^\lambda\right| < \varepsilon + \varepsilon\sqrt{\mathbb{E}_{\lambda,\omega}\left\{\frac{X_n^2}{n^2}\right\}} \tag{6.16}$$

for n large enough. We see that if $\lim_{n\to\infty}\mathbb{E}_{\lambda,\omega}\left\{\frac{X_n^2}{n^2}\right\}$ exists then so $\lim_{n\to\infty}\mathbb{E}_{\lambda,\omega}\left\{\frac{X_n}{n}\right\}$.

Proposition 6.2.1. *For almost all environment ω,*

$$\lim_{n\to\infty}\mathbb{E}_{\lambda,\omega}\left\{\frac{X_n^2}{n^2}\right\} = \frac{1}{[L_\lambda]^2}. \tag{6.17}$$

Proof. We consider a function $g_\lambda \ge 0$, defined on $\mathbb{Z}$, such that $(P_{\lambda,\omega} - I)g_\lambda \equiv f_\lambda$ and $g_\lambda(0) = 0$. For example, we can take

$$g_\lambda(m) = \begin{cases} \sum\limits_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)e^{2\ell\lambda}} \sum\limits_{s=-\infty}^{\ell} \pi(T^s\omega)e^{(2s-1)\lambda}f_\lambda(s), & if \quad m \ge 1 \\ 0, & if \quad m = 0 \\ -\sum\limits_{\ell=1}^{-m} \frac{1}{c(T^{-\ell}\omega)}e^{2\ell\lambda} \sum\limits_{s=-\infty}^{-\ell} \pi(T^s\omega)e^{(2s-1)\lambda}f_\lambda(s), & if \quad m \le -1 \end{cases}$$

then $(P_{\lambda,\omega}-I)g_\lambda(m) = f_\lambda(m)$ for any $m \in \mathbb{Z}$. Replacing m by X_n and take the expectation, one has

$$\mathbb{E}_{\lambda,\omega}\left\{g_\lambda(X_n)\right\} = \frac{n(n-1)}{2}, \quad \forall n \geq 0. \tag{6.18}$$

The formula (6.18) can be rewritten as

$$\mathbb{E}_\omega\left\{\frac{g_\lambda(X_n)}{X_n^2} \times \frac{X_n^2}{n^2}\right\} \sim \frac{1}{2}$$

and note that if $\lim_{m\to\infty} \frac{g_\lambda(m)}{m^2}$ exists then so $\lim_{n\to+\infty} \mathbb{E}_\omega\left\{\frac{X_n^2}{n^2}\right\}$.

The next step we will compute the limit of $\frac{g_\lambda(m)}{m^2}$ by using the lemmas 6.2.1 and 6.2.2.

Lemma 6.2.3. *With function g_λ defined as above*

$$\lim_{m\to\pm\infty} \frac{g_\lambda(m)}{m^2} = \frac{1}{2}\left[L_\lambda\right]^2. \tag{6.19}$$

Proof. Consider the case $m > 0$. Put

$$\begin{aligned}
\xi_1 &= \frac{1}{m^2}\sum_{\ell=0}^{m-1} \frac{\rho^\ell}{c(T^\ell\omega)}\sqrt{\rho}\sum_{s=-\infty}^{0} \pi(T^s\omega)\rho^{-s} f_\lambda(s),\\
\xi_2 &= \frac{1}{m^2}\sum_{\ell=0}^{m-1} \frac{\rho^\ell}{c(T^\ell\omega)}\sqrt{\rho}\sum_{s=1}^{\ell} \pi(T^s\omega)\rho^{-s} f_\lambda(s),\\
\xi_3 &= \frac{1}{m^2}\sum_{\ell=0}^{m-1} \frac{\rho^\ell}{c(T^\ell\omega)}\sqrt{\rho}\sum_{s=1}^{\ell} \pi(T^s\omega)\rho^{-s} s.
\end{aligned}$$

By the definition of function g_λ, we have $\dfrac{g_\lambda(m)}{m^2} = \xi_1 + \xi_2$. We will prove that

$$\lim_{m\to+\infty} \xi_1 = 0 \tag{6.20}$$

and

$$\lim_{m\to+\infty} \xi_2 = \frac{1}{2}\left[L_\lambda\right]^2. \tag{6.21}$$

By (6.11) and $\lim\limits_{s\to\infty} \dfrac{f_\lambda(s)}{s} = L_\lambda$ then $\sum\limits_{s=-\infty}^{0} \pi(T^s\omega)\rho^{-s} f_\lambda(s)$ is bounded which completes (6.20).

Proof of (6.21). Replacing $\ell - s$ by k we obtain

$$\begin{aligned}
\xi_3 &= \frac{1}{m^2}\sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)}\sqrt{\rho}\sum_{s=1}^{\ell} \pi(T^s\omega)\rho^{\ell-s} s = \frac{1}{m^2}\sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)}\sqrt{\rho}\sum_{k=0}^{\ell-1} \pi(T^{-k}\omega)\rho^k(\ell-k)\\
&= \frac{1}{m^2}\sum_{\ell=0}^{m-1} \frac{\ell}{c(T^\ell\omega)}\left[\sqrt{\rho}\sum_{k=0}^{\ell-1} \pi(T^{-k}\omega)\rho^k\right]\circ T^\ell - \frac{1}{m^2}\sum_{\ell=0}^{m-1} \frac{\sqrt{\rho}}{c(T^\ell\omega)}\left[\sum_{k=0}^{\ell-1} \pi(T^{-k}\omega)k\rho^k\right]\circ T^\ell.
\end{aligned}$$

Since $\sqrt{\rho}\sum_{k=0}^{\ell-1}\pi(T^{-k}\omega)k\rho^k$ is bounded by (6.11) then

$$\lim_{m\to+\infty}\frac{1}{m^2}\sum_{\ell=0}^{m-1}\frac{\sqrt{\rho}}{c(T^\ell\omega)}\left[\sum_{k=0}^{\ell-1}\pi(T^{-k}\omega)k\rho^k\right]\circ T^\ell = 0.$$

On the other hand, since $\lim_{\ell\to+\infty}\sqrt{\rho}\sum_{k=0}^{\ell-1}\pi(T^{-k}\omega)\rho^k = H_\lambda(\omega)$ then

$$\lim_{m\to+\infty}\sup_{\ell\le m}\frac{1}{m}\left|\sqrt{\rho}\sum_{k=0}^{\ell-1}\pi(T^{-k}\omega)\rho^k - H_\lambda(\omega)\right| = 0.$$

And hence,

$$\lim_{m\to+\infty}\xi_3 = \lim_{m\to+\infty}\frac{1}{m}\sum_{\ell=0}^{m-1}\frac{H_\lambda}{c}\circ T^\ell(\omega)\left(\frac{\ell}{m}\right) = \frac{1}{2}L_\lambda.$$

Moreover, since $\lim_{s\to\infty}\frac{f_\lambda(s)}{s} = L_\lambda$ then $\lim_{m\to+\infty}\sup_{s\le m}\frac{1}{m}|f_\lambda(s) - sL_\lambda| = 0$. It follows that

$$\lim_{m\to+\infty}\xi_2 = \lim_{m\to+\infty}\xi_3 L_\lambda = \frac{1}{2}[L_\lambda]^2$$

which completes (6.21).

Similarly, we get also the same result for the case $m < 0$. □

By lemma 6.2.3, for any $\varepsilon' > 0$, there exists $M' > 0$ such that for any $|m| > M'$ then

$$\left|\frac{m^2}{g_\lambda(m)} - \frac{2}{[L_\lambda]^2}\right| < \varepsilon'/2.$$

Put

$$II_1^\lambda = \mathbb{E}_{\lambda,\omega}\left\{\frac{X_n^2}{n^2}\mathbb{1}_{\{|X_n|\le M'\}}\right\} - \mathbb{E}_{\lambda,\omega}\left\{\frac{2}{[L_\lambda]^2}\frac{g_\lambda(X_n)}{n(n-1)}\mathbb{1}_{\{|X_n|\le M'\}}\right\}$$
$$II_2^\lambda = \mathbb{E}_{\lambda,\omega}\left\{\frac{X_n^2}{n^2}\mathbb{1}_{\{|X_n|> M'\}}\right\} - \mathbb{E}_{\lambda,\omega}\left\{\frac{2}{[L_\lambda]^2}\frac{g_\lambda(X_n)}{n(n-1)}\mathbb{1}_{\{|X_n|> M'\}}\right\}$$

then

$$\begin{aligned}\left|II_1^\lambda\right| &= \left|\frac{1}{n^2}\mathbb{E}_{\lambda,\omega}\left\{\left[X_n^2 - \frac{2g_\lambda(X_n)}{[L_\lambda]^2}\frac{n^2}{n(n-1)}\right]\mathbb{1}_{\{|X_n|\le M'\}}\right\}\right| \\ &\le \frac{1}{n^2}\mathbb{E}_{\lambda,\omega}\left\{\left|X_n^2 - \frac{2g_\lambda(X_n)}{[L_\lambda]^2}\frac{n^2}{n(n-1)}\right|\mathbb{1}_{\{|X_n|\le M'\}}\right\} \\ &< \varepsilon'/2\end{aligned}$$

and

$$\left|II_2^\lambda\right| = \left|\mathbb{E}_{\lambda,\omega}\left\{\left(\frac{X_n^2}{n^2} - \frac{2}{[L_\lambda]^2}\frac{g_\lambda(X_n)}{n(n-1)}\right)\mathbb{1}_{\{|X_n|>M'\}}\right\}\right|$$

$$
\begin{aligned}
&= \left|\mathbb{E}_{\lambda,\omega}\left\{\frac{g_\lambda(X_n)}{n(n-1)}\left(\frac{n(n-1)}{n^2}\frac{X_n^2}{g_\lambda(X_n)}-\frac{2}{[L_\lambda]^2}\right)\mathbb{1}_{\{|X_n|>M'\}}\right\}\right| \\
&< \varepsilon'/2
\end{aligned}
$$

for n large enough since $g(m) \geq 0$ for any $m \in \mathbb{Z}$. It follows that

$$
\left|\mathbb{E}_{\lambda,\omega}\left\{\frac{X_n^2}{n^2}\right\}-\frac{1}{[L_\lambda]^2}\right| = \left|II_1^\lambda + II_2^\lambda\right| \leq \left|II_1^\lambda\right| + \left|II_2^\lambda\right| < \varepsilon'
$$

for n large enough. □

We have thus proved that $\lim_{n\to\infty}\mathbb{E}_{\lambda,\omega}\left\{\frac{X_n^2}{n^2}\right\} = \frac{1}{[L_\lambda]^2}$. From (6.16), we obtain

$$
\lim_{n\to\infty}\mathbb{E}_{\lambda,\omega}\left\{\frac{X_n}{n}\right\} = \frac{1}{L_\lambda} = \left[\int_\Omega \frac{H_\lambda}{c}d\mu\right]^{-1} = d(\lambda).
$$

Finally, by (6.13) one has

$$
\begin{aligned}
\lim_{\lambda\to0^+}\frac{d(\lambda)}{\lambda} &= \lim_{\lambda\to0^+}\frac{(1-e^{-2\lambda})}{\lambda}\left[\int_\Omega\frac{(1-e^{-2\lambda})H_\lambda}{c}d\mu\right]^{-1} \\
&= \lim_{\lambda\to0^+}\frac{(1-e^{-2\lambda})}{\lambda(e^{-\lambda}+e^{\lambda})}\left[\int_\Omega c\,d\mu\int_\Omega\frac{1}{c}\,d\mu\right]^{-1} \\
&= \left[\int_\Omega c\,d\mu\int_\Omega\frac{1}{c}\,d\mu\right]^{-1}.
\end{aligned}
$$

□

Theorem 6.2.3. *For almost all environment ω and for $\lambda < 0$*

$$
\lim_{\lambda\to0^-}\frac{d(\lambda)}{\lambda} = \lim_{\lambda\to0^-}\frac{1}{\lambda}\lim_{n\to\infty}\mathbb{E}_{\lambda,\omega}\left\{\frac{X_n}{n}\right\} = \left[\int_\Omega c\,d\mu\int_\Omega\frac{1}{c}\,d\mu\right]^{-1}. \tag{6.22}
$$

Proof. The proof of this theorem is very similar to theorem 6.2.2 which modifies functions f_λ and g_λ, defined on $\mathbb{Z}$, as follows

$$
f_\lambda(m) = \begin{cases}
-\sum_{\ell=0}^{m-1}\frac{1}{c(T^\ell\omega)e^{2\ell\lambda}}\sum_{s=\ell}^{+\infty}\pi(T^s\omega)e^{(2s+1)\lambda}, & if \quad m \geq 1 \\
0, & if \quad m = 0 \\
\sum_{\ell=1}^{-m}\frac{1}{c(T^{-\ell}\omega)}e^{2\ell\lambda}\sum_{s=-\ell}^{+\infty}\pi(T^s\omega)e^{(2s+1)\lambda}, & if \quad m \leq -1
\end{cases}
$$

and

$$
g_\lambda(m) = \begin{cases}
-\sum_{\ell=0}^{m-1}\frac{1}{c(T^\ell\omega)e^{2\ell\lambda}}\sum_{s=\ell}^{+\infty}\pi(T^s\omega)e^{(2s+1)\lambda}f_\lambda(s), & if \quad m \geq 1 \\
0, & if \quad m = 0 \\
\sum_{\ell=1}^{-m}\frac{1}{c(T^{-\ell}\omega)}e^{2\ell\lambda}\sum_{s=-\ell}^{+\infty}\pi(T^s\omega)e^{(2s+1)\lambda}f_\lambda(s), & if \quad m \leq -1
\end{cases}
$$

where ω is fixed. □

***Remark* 6.2.1.** *We have proved that for $\lambda \neq 0$ and for almost all ω*

$$\lim_{n\to\infty} \mathbb{E}_{\lambda,\omega}\left\{\frac{X_n}{n}\right\} = d(\lambda) \tag{6.23}$$

and

$$\lim_{n\to\infty} \mathbb{E}_{\lambda,\omega}\left\{\frac{X_n^2}{n^2}\right\} = d(\lambda)^2. \tag{6.24}$$

This implies that for ω a.s

$$\frac{X_n}{n} \xrightarrow{\ P\ } d(\lambda) \quad \text{as } n \longrightarrow \infty \tag{6.25}$$

where $\xrightarrow{P}$ is denoted as the convergence in probability.

6.3 Markov processes in Random environment with a drift

We consider Markov process $(X_t)_{t\in\mathbb{R}}$ on $\mathbb{Z}$ with $X_0 = 0$, the generator infinitesimal

$$L_{\lambda,\omega}f(k) = e^{-\lambda}c(T^{k-1}\omega)f(k-1) + e^{\lambda}c(T^k\omega)f(k+1) - \pi(T^k\omega)f(k), \tag{6.26}$$

where $\pi = e^{\lambda}c + e^{-\lambda}c \circ T^{-1}$.

Theorem 6.3.1. *For almost all environment ω,*

$$\lim_{\lambda\to 0} \frac{d(\lambda)}{\lambda} = \lim_{\lambda\to 0} \frac{1}{\lambda} \lim_{t\to+\infty} \mathbb{E}_{\lambda,\omega}\left\{\frac{X_t}{t}\right\} = 2\left[\int_{\Omega} \frac{1}{c} d\mu\right]^{-1} \tag{6.27}$$

if $c^{-1} \in L^1(\mu)$.

Proof. This theorem is proved by Theorems 6.3.2 and 6.3.3. □

Theorem 6.3.2. *For almost all environment ω and for $\lambda > 0$*

$$\lim_{\lambda\to 0^+} \frac{d_\omega(\lambda)}{\lambda} = \lim_{\lambda\to 0^+} \frac{1}{\lambda} \lim_{t\to\infty} \mathbb{E}_{\lambda,\omega}\left\{\frac{X_t}{t}\right\} = 2\left[\int_{\Omega} \frac{1}{c}\, d\mu\right]^{-1}. \tag{6.28}$$

Proof. Fix $\omega \in \Omega$. We consider a functions f_λ, defined on $\mathbb{Z}$, such that $L_{\lambda,\omega}f_\lambda \equiv 1$ and $f_\lambda(0) = 0$. For example, we can take

$$f_\lambda(m) = \begin{cases} \sum\limits_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)e^{2\ell\lambda}} \sum\limits_{s=-\infty}^{\ell} e^{(2s-1)\lambda}, & if \quad m \geq 1 \\ 0, & if \quad m = 0 \\ -\sum\limits_{\ell=1}^{-m} \frac{1}{c(T^{-\ell}\omega)} e^{2\ell\lambda} \sum\limits_{s=-\infty}^{-\ell} e^{(2s-1)\lambda}, & if \quad m \leq -1 \end{cases}$$

It is easy to check that $L_{\lambda,\omega}f_\lambda(m) = 1$ for any $m \in \mathbb{Z}$. Replacing m by X_t and take the expectation, one has

$$\mathbb{E}_{\lambda,\omega}\left\{f_\lambda(X_t)\right\} = t \quad \forall t \geq 0. \tag{6.29}$$

The formula (6.29) can be rewritten as

$$\mathbb{E}_\omega \left\{ \frac{f_\lambda(X_t)}{X_t} \times \frac{X_t}{t} \right\} = 1$$

and note that if $\lim_{m\to\infty} \frac{f_\lambda(m)}{m}$ exists then so $\lim_{n\to+\infty} \mathbb{E}_\omega \left\{\frac{X_t}{t}\right\}$.

The next step we will compute the limit of $\frac{f_\lambda(m)}{m}$ by using the pointwise ergodic theorem.

Lemma 6.3.1. *Put* $\rho = \dfrac{1}{e^{2\lambda}}$ *and function* f_λ *defined as above, one has*

$$\lim_{m\to\pm\infty} \frac{f_\lambda(m)}{m} = \frac{\sqrt{\rho}}{1-\rho} \int_\Omega \frac{1}{c} d\mu = L_\lambda. \tag{6.30}$$

Proof. By the definition of function f_λ, for $m > 0$

$$\begin{aligned} \frac{f_\lambda(m)}{m} &= \frac{1}{m} \sum_{\ell=0}^{m-1} \frac{\rho^\ell}{c(T^\ell\omega)} \sqrt{\rho} \sum_{s=-\infty}^{\ell} \rho^{-s} = \frac{1}{m} \sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)} \sqrt{\rho} \sum_{s=-\infty}^{\ell} \rho^{\ell-s} \\ &= \frac{1}{m} \sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)} \sqrt{\rho} \sum_{k=0}^{+\infty} \rho^k = \frac{\sqrt{\rho}}{1-\rho} \frac{1}{m} \sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)} \end{aligned}$$

and hence by pointwise ergodic theorem (6.30) is followed.

Similarly for $m < 0$ we will obtain the desired result. □

For any $\varepsilon > 0$, by (6.30) there exists $M > 0$ such that for any $|m| > M$ then

$$\left| \frac{1}{L_\lambda} \frac{f_\lambda(m)}{m} - 1 \right| < \varepsilon. \tag{6.31}$$

We now combine (6.29) and (6.31) to compute the limit of $\mathbb{E}_{\lambda,\omega} \left\{\frac{X_t}{t}\right\}$. Put

$$I_1^\lambda = \mathbb{E}_{\lambda,\omega} \left\{ \frac{X_t}{t} \mathbb{1}_{\{|X_t|\le M\}} \right\} - \mathbb{E}_{\lambda,\omega} \left\{ \frac{1}{L_\lambda} \frac{f_\lambda(X_t)}{t} \mathbb{1}_{\{|X_t|\le M\}} \right\},$$

$$I_2^\lambda = \mathbb{E}_{\lambda,\omega} \left\{ \frac{X_t}{t} \mathbb{1}_{\{|X_t|> M\}} \right\} - \mathbb{E}_{\lambda,\omega} \left\{ \frac{1}{L_\lambda} \frac{f_\lambda(X_t)}{t} \mathbb{1}_{\{|X_t|> M\}} \right\}$$

then

$$\begin{aligned} \left|I_1^\lambda\right| &= \left| \frac{1}{t} \mathbb{E}_{\lambda,\omega} \left\{ \left[X_t - \frac{f_\lambda(X_t)}{L_\lambda} \right] \mathbb{1}_{\{|X_t|\le M\}} \right\} \right| \\ &\le \frac{1}{t} \mathbb{E}_{\lambda,\omega} \left\{ \left| X_t - \frac{f_\lambda(X_t)}{L_\lambda} \right| \mathbb{1}_{\{|X_t|\le M\}} \right\} \\ &< \varepsilon \end{aligned}$$

and

$$\left|I_2^\lambda\right| = \left| \mathbb{E}_{\lambda,\omega} \left\{ \left(1 - \frac{1}{L_\lambda} \frac{f_\lambda(X_t)}{X_t} \right) \frac{X_t}{t} \mathbb{1}_{\{|X_t|> M\}} \right\} \right|$$

$$< \quad \varepsilon\sqrt{\mathbb{E}_{\lambda,\omega}\left\{\frac{X_t^2}{t^2}\right\}}$$

for t large enough. It follows that

$$\left|\mathbb{E}_{\lambda,\omega}\left\{\frac{X_t}{t}\right\}-\frac{1}{L_\lambda}\right| = \left|I_1^\lambda + I_2^\lambda\right| \le \left|I_1^\lambda\right| + \left|I_2^\lambda\right| < \varepsilon + \varepsilon\sqrt{\mathbb{E}_{\lambda,\omega}\left\{\frac{X_t^2}{t^2}\right\}} \tag{6.32}$$

for t large enough. We see that if $\lim_{t\to+\infty}\mathbb{E}_{\lambda,\omega}\left\{\frac{X_t^2}{t^2}\right\}$ exists then so $\lim_{t\to+\infty}\mathbb{E}_{\lambda,\omega}\left\{\frac{X_t}{t}\right\}$.

Proposition 6.3.1. *For almost all environment ω,*

$$\lim_{t\to+\infty}\mathbb{E}_{\lambda,\omega}\left\{\frac{X_t^2}{t^2}\right\} = \frac{1}{[L_\lambda]^2}. \tag{6.33}$$

Proof. We consider a function $g_\lambda \ge 0$, defined on $\mathbb{Z}$, such that $L_{\lambda,\omega}g_\lambda \equiv f_\lambda$ and $g_\lambda(0)=0$. For example, we can take

$$g_\lambda(m) = \begin{cases} \sum\limits_{\ell=0}^{m-1}\frac{1}{c(T^\ell\omega)e^{2\ell\lambda}}\sum\limits_{s=-\infty}^{\ell} e^{(2s-1)\lambda}f_\lambda(s), & if \quad m\ge 1\\ 0, & if \quad m=0\\ -\sum\limits_{\ell=1}^{-m}\frac{1}{c(T^{-\ell}\omega)}e^{2\ell\lambda}\sum\limits_{s=-\infty}^{-\ell} e^{(2s-1)\lambda}f_\lambda(s), & if \quad m\le -1\end{cases}$$

then $L_{\lambda,\omega}g(m) = f(m)$ for any $m\in\mathbb{Z}$. Replacing m by X_t and take the expectation, one has

$$\mathbb{E}_{\lambda,\omega}\{g(X_t)\} = \frac{t^2}{2}, \quad \forall t\ge 0. \tag{6.34}$$

The formula (6.34) can be rewritten as

$$\mathbb{E}_\omega\left\{\frac{g_\lambda(X_t)}{X_t^2}\times\frac{X_t^2}{t^2}\right\} = \frac{1}{2}$$

and note that if $\lim_{m\to\infty}\frac{g_\lambda(m)}{m^2}$ exists then so $\lim_{n\to+\infty}\mathbb{E}_\omega\left\{\frac{X_n}{n}\right\}$.

The next step we will compute the limit of $\frac{g_\lambda(m)}{m^2}$ by using the pointwise ergodic theorem and lemma 5.1.1.

Lemma 6.3.2. *With function g_λ defined as above*

$$\lim_{m\to\pm\infty}\frac{g_\lambda(m)}{m^2} = \frac{1}{2}[L_\lambda]^2. \tag{6.35}$$

Proof. Consider the case $m>0$. Put

$$\xi_1 \quad = \quad \frac{1}{m^2}\sum_{\ell=0}^{m-1}\frac{\rho^\ell}{c(T^\ell\omega)}\sqrt{\rho}\sum_{s=-\infty}^{0}\rho^{-s}f_\lambda(s),$$

$$\begin{aligned}
\xi_2 &= \frac{1}{m^2}\sum_{\ell=0}^{m-1}\frac{\rho^\ell}{c(T^\ell\omega)}\sqrt{\rho}\sum_{s=1}^{\ell}\rho^{-s}f_\lambda(s),\\
\xi_3 &= \frac{1}{m^2}\sum_{\ell=0}^{m-1}\frac{\rho^\ell}{c(T^\ell\omega)}\sqrt{\rho}\sum_{s=1}^{\ell}\rho^{-s}s.
\end{aligned}$$

By the definition of function g_λ, we have $\dfrac{g_\lambda(m)}{m^2} = \xi_1 + \xi_2$. We will prove that

$$\lim_{m\to+\infty}\xi_1 = 0 \tag{6.36}$$

and

$$\lim_{m\to+\infty}\xi_2 = \frac{1}{2}\left[L_\lambda\right]^2 \tag{6.37}$$

By (6.11) and $\lim\limits_{s\to\infty}\dfrac{f_\lambda(s)}{s} = L_\lambda$ then $\sum\limits_{s=-\infty}^{0}\rho^{-s}f_\lambda(s)$ is bounded which completes (6.36).

Proof of (6.37). Replacing $\ell - s$ by k we obtain

$$\begin{aligned}
\xi_3 &= \frac{1}{m^2}\sum_{\ell=0}^{m-1}\frac{1}{c(T^\ell\omega)}\sqrt{\rho}\sum_{s=1}^{\ell}\rho^{\ell-s}s = \frac{1}{m^2}\sum_{\ell=0}^{m-1}\frac{1}{c(T^\ell\omega)}\sqrt{\rho}\sum_{k=0}^{\ell-1}\rho^k(\ell-k)\\
&= \frac{1}{m^2}\sum_{\ell=0}^{m-1}\frac{\ell}{c(T^\ell\omega)}\sqrt{\rho}\sum_{k=0}^{\ell-1}\rho^k - \frac{1}{m^2}\sum_{\ell=0}^{m-1}\frac{\sqrt{\rho}}{c(T^\ell\omega)}\sum_{k=0}^{\ell-1}k\rho^k.
\end{aligned}$$

Since $\sqrt{\rho}\sum\limits_{k=0}^{\ell-1}\pi(T^{-k}\omega)k\rho^k$ is bounded by (6.11) and $\lim\limits_{\ell\to+\infty}\sqrt{\rho}\sum\limits_{k=0}^{\ell-1}\rho^k = \dfrac{\sqrt{\rho}}{1-\rho}$ then by lemma 5.1.1, one has

$$\lim_{m\to+\infty}\xi_3 = \frac{1}{2}\int_\Omega\frac{1}{c}d\mu\frac{\sqrt{\rho}}{1-\rho} = \frac{1}{2}L_\lambda$$

Moreover, since $\lim\limits_{s\to\infty}\dfrac{f_\lambda(s)}{s} = L_\lambda$ then $\lim\limits_{m\to\infty}\sup\limits_{s\le m}\dfrac{1}{m}\left|f_\lambda(s) - sL_\lambda\right| = 0$. It follows that

$$\lim_{m\to+\infty}\xi_2 = \lim_{m\to+\infty}\xi_3 L_\lambda = \frac{1}{2}\left[L_\lambda\right]^2$$

which completes (6.37).

Similarly, we get also the same result for the case $m < 0$. $\square$

For any $\varepsilon' > 0$, by (6.35) there exists $M' > 0$ such that for any $m > M'$ then

$$\left|\frac{m^2}{g_\lambda(m)} - \frac{2}{\left[L_\lambda\right]^2}\right| < \varepsilon'/2. \tag{6.38}$$

We now combine (6.34) and (6.38) to compute $\lim_{t\to+\infty}\mathbb{E}_{\lambda,\omega}\left\{\frac{X_t^2}{t^2}\right\}$. Put

$$II_1^\lambda = \mathbb{E}_{\lambda,\omega}\left\{\frac{X_t^2}{t^2}\mathbb{1}_{\{|X_t|\le M'\}}\right\} - \mathbb{E}_{\lambda,\omega}\left\{\frac{2}{\left[L_\lambda\right]^2}\frac{g_\lambda(X_t)}{t^2}\mathbb{1}_{\{|X_t|\le M'\}}\right\}$$

$$II_2^\lambda = \mathbb{E}_{\lambda,\omega}\left\{\frac{X_t^2}{t^2}\mathbb{1}_{\{|X_t|>M'\}}\right\} - \mathbb{E}_{\lambda,\omega}\left\{\frac{2}{[L_\lambda]^2}\frac{g_\lambda(X_t)}{t^2}\mathbb{1}_{\{|X_t|>M'\}}\right\}$$

then

$$\begin{aligned}\left|II_1^\lambda\right| &= \left|\frac{1}{t^2}\mathbb{E}_{\lambda,\omega}\left\{\left[X_t^2 - \frac{2g_\lambda(X_t)}{[L_\lambda]^2}\right]\mathbb{1}_{\{|X_t|\le M'\}}\right\}\right| \\ &\le \frac{1}{t^2}\mathbb{E}_{\lambda,\omega}\left\{\left|X_t^2 - \frac{2g_\lambda(X_t)}{[L_\lambda]^2}\right|\mathbb{1}_{\{|X_t|\le M'\}}\right\} \\ &< \varepsilon'/2\end{aligned}$$

and

$$\begin{aligned}\left|II_2^\lambda\right| &= \left|\mathbb{E}_{\lambda,\omega}\left\{\left(\frac{X_t^2}{t^2} - \frac{2}{[L_\lambda]^2}\frac{g_\lambda(X_t)}{t^2}\right)\mathbb{1}_{\{|X_t|>M'\}}\right\}\right| \\ &\le \mathbb{E}_{\lambda,\omega}\left\{\frac{g_\lambda(X_t)}{t^2}\left|\frac{X_t^2}{g_\lambda(X_t)} - \frac{2}{[L_\lambda]^2}\right|\mathbb{1}_{\{|X_t|>M'\}}\right\} \\ &< \varepsilon'/2\end{aligned}$$

for n large enough. It follows that

$$\left|\mathbb{E}_{\lambda,\omega}\left\{\frac{X_t^2}{t^2}\right\} - \frac{1}{[L_\lambda]^2}\right| = \left|II_1^\lambda + II_2^\lambda\right| \le \left|II_1^\lambda\right| + \left|II_2^\lambda\right| < \varepsilon'$$

for n large enough. □

We have thus proved that $\lim_{t\to+\infty}\mathbb{E}_{\lambda,\omega}\left\{\frac{X_t^2}{t^2}\right\} = \frac{1}{[L_\lambda]^2}$. From (6.32) we obtain

$$\lim_{t\to+\infty}\mathbb{E}_{\lambda,\omega}\left\{\frac{X_t}{t}\right\} = \frac{1}{L_\lambda} = \frac{1-\rho}{\sqrt{\rho}}\left[\int_\Omega \frac{1}{c}d\mu\right]^{-1} = (e^\lambda - e^{-\lambda})\left[\int_\Omega \frac{1}{c}d\mu\right]^{-1} = d(\lambda).$$

Finally

$$\lim_{\lambda\to0^+}\frac{d(\lambda)}{\lambda} = \lim_{\lambda\to0^+}\frac{e^\lambda - e^{-\lambda}}{\lambda}\left[\int_\Omega \frac{1}{c}d\mu\right]^{-1} = 2\left[\int_\Omega \frac{1}{c}d\mu\right]^{-1}.$$

□

Theorem 6.3.3. *For almost all environment ω and for $\lambda < 0$*

$$\lim_{\lambda\to0^-}\frac{d(\lambda)}{\lambda} = \lim_{\lambda\to0^-}\frac{1}{\lambda}\lim_{t\to+\infty}\mathbb{E}_{\lambda,\omega}\left\{\frac{X_t}{t}\right\} = 2\left[\int_\Omega \frac{1}{c}d\mu\right]^{-1}. \tag{6.39}$$

Proof. The proof of this theorem is very similar to theorem 6.2.2 which modifies functions f_λ and g_λ, defined on $\mathbb{Z}$, as follows

$$f_\lambda(m) = \begin{cases} -\sum_{\ell=0}^{m-1}\frac{1}{c(T^\ell\omega)e^{2\ell\lambda}}\sum_{s=\ell}^{+\infty}e^{(2s+1)\lambda}, & if \quad m\ge1 \\ 0, & if \quad m=0 \\ \sum_{\ell=1}^{-m}\frac{1}{c(T^{-\ell}\omega)}e^{2\ell\lambda}\sum_{s=-\ell}^{+\infty}e^{(2s+1)\lambda}, & if \quad m\le-1\end{cases}$$

and

$$g_\lambda(m) = \begin{cases} -\sum_{\ell=0}^{m-1} \frac{1}{c(T^\ell\omega)e^{2\ell\lambda}} \sum_{s=\ell}^{+\infty} e^{(2s+1)\lambda} f_\lambda(s), & if \quad m \geq 1 \\ 0, & if \quad m = 0 \\ \sum_{\ell=1}^{-m} \frac{1}{c(T^{-\ell}\omega)} e^{2\ell\lambda} \sum_{s=-\ell}^{+\infty} e^{(2s+1)\lambda} f_\lambda(s), & if \quad m \leq -1 \end{cases}$$

where ω is fixed. □

***Remark* 6.3.1.** *We have proved that for $\lambda \neq 0$ and for almost all ω*

$$\lim_{t \to +\infty} \mathbb{E}_{\lambda,\omega} \left\{ \frac{X_t}{t} \right\} = d(\lambda) \tag{6.40}$$

and

$$\lim_{t \to +\infty} \mathbb{E}_{\lambda,\omega} \left\{ \frac{X_t^2}{t^2} \right\} = d(\lambda)^2 \tag{6.41}$$

with $d(\lambda) = (e^\lambda - e^{-\lambda}) \left[\int_\Omega \frac{1}{c} d\mu\right]^{-1}$. This implies that for ω a.s

$$\frac{X_t}{t} \xrightarrow{\quad P \quad} d(\lambda) \quad as\ t \longrightarrow +\infty \tag{6.42}$$

where $\xrightarrow{P}$ is denoted as the convergence in probability.

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Glossaire

- $\mathbb{P}\{A\}$: probability of an event A
- $\mathbb{E}\{X\}$: mathematical expectation of random variable X
- $\xrightarrow{D}$: converges in distribution
- μ $a.s$: almost surely under measure μ
- $\mathbb{1}_{\{\}}$: indicator function.